**Stat Mech & Scalings**

*31 August 2011*

**Goals**
- Refresh your memory about statistical mechanics
- Practice scaling arguments

**Statistical Mechanics**

1. Suppose we have a non-relativistic, classical, ideal gas with a constant temperature $T$. These particles obey [Maxwell-Boltzmann statistics](https://en.wikipedia.org/wiki/Maxwell-Boltzmann_statistics).

(a) Roughly, what do each of the qualifiers “non-relativistic”, “classical”, and “ideal” mean?

**Solution:**
- non-relativistic: velocities much less than the speed of light
- classical: no quantum effects like spin
- ideal: point particles that don’t interact strongly

(b) How does the probability of a particle being in a particular state with energy $E$ scale with $E$ and/or $T$?

**Solution:** This is the Boltzmann factor

$$p(E) \propto e^{-E/kT}.$$  

You can derive it from the canonical ensemble, but I was just expecting you to remember it.

(c) What is the energy of a particle in a non-relativistic, classical, ideal gas?

**Solution:** The energy of a particle is just the kinetic energy

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}.$$  

(d) Suppose we have a particle with energy $E$. Sketch where it would be located in momentum space.

**Solution:** In momentum space, this is a sphere of radius

$$p = \sqrt{2mE}.$$  

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You may have seen most of this before, but key points from statistical mechanics will be important when talking about stellar structure, so it is worth going over again.

Don’t worry about constants. The right proportionality is what is important.

When I say momentum space, I mean a plot where the values along the $x, y, z$ axes are the values of $p_x, p_y, p_z.$
(e) What if we have a particle with energy between $E$ and $E + \Delta E$? What is the volume of the region in momentum space where it could be?

**Solution:** In momentum space, this is a spherical shell.

The volume of a spherical shell is $4\pi r^2 dr$. The radius of our shell in momentum space is $\sqrt{2mE}$. The thickness of our shell in energy is $\Delta E$. How thick is this in momentum space? $\Delta p = \sqrt{2m(\Delta E/(2\sqrt{E}))}$. So, if we’re ruthless with the constants,

$$\text{Volume} \propto m^{3/2} E^{1/2} e^{-E/kT} \Delta E.$$  

(f) Previously, you wrote down the probability of a particle being in a particular state $E$. Assume that the number of states with energy between $E$ and $E + \Delta E$ is proportional to the volume in momentum space you just found. What is the probability of a particle being in any state with energy between $E$ and $E + \Delta E$?

**Solution:** The probability of a particle being in any state with energy between $E$ and $E + \Delta E$ is the product of the prob-
ability of a particle being in a particular state with energy \( E \) times the number of states with that energy. 

\[
\text{Probability} \propto \frac{m^{3/2}E^{1/2}\Delta E}{S} \times e^{-E/kT}
\]

where \( S \) is some constant with units of momentum\(^3\) that characterizes the size of a single state. The constant will vanish in the next step, but \( S \) makes sure that the expression we wrote down is dimensionless, as a probability should be.

(g) Your previous answer should have been in the form \( f(E)\Delta E \). This is a **probability distribution function**, which represents the probability of finding a particle with energy between \( E \) and \( E + \Delta E \). To make this an equality rather than a proportionality, we would need to integrate this from \( E = 0 \) to \( E = \infty \) and normalize such that it is equal to one.

\[
\int_0^\infty f(E) dE = 1.
\]

Don’t do the integral, but keep in mind that a probability distribution integrated over all possibilities must always equal one. The answer you would find is

\[
f(E)dE = \frac{2}{kT} \left( \frac{E}{\pi kT} \right)^{1/2} e^{-E/kT} dE
\]

Sketch this function and make a guess where the peak is.

**Solution:**

\[
\begin{array}{c}
\begin{array}{c}
\text{f(E)}
\end{array}
\end{array}
\]

The peak, which corresponds to the most probable energy, is at \( E = \frac{1}{2}kT \). (The long tail means that the mean value of the energy is actually \( E = \frac{3}{2}kT \), giving you the result you may remember for the ideal gas.)

(h) Suppose we wanted to do something similar for photons. Which of our initial assumptions aren’t valid anymore? What sorts of things would we have to do differently?
Solution:

- Relativistic: \( v = c, E = pc \)
- Quantum: Photons have spin-1

So, for example, when we wrote down the probability for a particle to be in a particular state, we would need to use Bose-Einstein statistics.
Scaling Arguments

2. In lecture on Tuesday, we wrote down the equation of hydrostatic equilibrium
\[
\frac{dP}{dr} = -g\rho
\]
and applied it to an isothermal atmosphere.

(a) What do we mean when we say ‘scale height’?

**Solution:** The scale height is the distance over which a quantity such as the pressure or density changes appreciably (usually by a factor of \(e\)).

(b) Use your definition of ‘scale height’ to make an approximation for the value of \(\frac{dP}{dr}\).

**Solution:** If we increase \(r\) by a scale height \(\Delta r \sim h\), then the pressure falls by \(1/e\), and \((1/e - 1) \approx -1\), so \(\Delta P \sim -P\). Therefore, we can write
\[
\frac{dP}{dr} \sim \frac{P}{h}
\]

(c) Use your approximation to solve an algebraic expression for the scale height, assuming that the pressure is gas pressure.

**Solution:** Rearranging,
\[
h \sim \frac{nkT}{\rho g} \sim \frac{kT}{mg}
\]
which should look familiar from lecture.

(d) Write your expression for the scale height as a scaling relation, normalized to values appropriate for the earth’s atmosphere.

**Solution:** \(k_b \approx 10^{-16}\) ergs/K, \(T \approx 300\) K, \(m \approx 2 \times 10^{-23}\) g,
\(g \approx 10^3\) cm/s^2
\[
h \sim 1.5 \times 10^6\ \text{cm} \left(\frac{T_{\text{atm}}}{300\ \text{K}}\right) \left(\frac{m}{2 \times 10^{-23}\ \text{g}}\right)^{-1} \left(\frac{g}{10^3\ \text{cm}/\text{s}^2}\right)^{-1}.
\]

(e) What if the pressure in the earth’s atmosphere were radiation pressure instead of gas pressure?

**Solution:**
\[
h \sim \frac{aT^4}{\rho g}
\]
The density of the Earth's atmosphere is \( \rho \sim 10^{-3} \text{ g cm}^{-3} \).
\[
h \sim 10^{-4} \text{ cm} \left( \frac{T_{\text{atm}}}{300 \text{ K}} \right)^4 \left( \frac{\rho}{10^{-3} \text{ g cm}^{-3}} \right)^{-1} \left( \frac{g}{10^3 \text{ cm/s}^2} \right)^{-1}.
\]