Molecular Weight & Energy Transport

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Goals
• Review mean molecular weight
• Practice working with diffusion

Mean Molecular Weight

1. We will frequently use $\mu$, $\mu_e$, and $\mu_I$ (the mean molecular weight per particle, per free electron, and per ion, respectively). Let’s practice computing these values for some common compositions.

Solution: Recall the definitions of $\mu_I$, $\mu_e$, and $\mu$:

$$\frac{1}{\mu_I} = \sum \frac{X_i}{A_i}$$

$$\frac{1}{\mu_e} = \sum \frac{Z_i X_i}{A_i}$$

$$\frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}$$

where $X_i$ is the mass fraction, $Z_i$ the charge, and $A_i$ the atomic number of a species $i$.

(a) 100% hydrogen (neutral)

Solution:

$$\frac{1}{\mu_I} = \frac{1}{1} = 1$$

$$\frac{1}{\mu_e} = 0$$

$$\frac{1}{\mu} = 1,$$

thus $\mu = 1$, $\mu_I = 1$, and $\mu_e$ is undefined (no free electrons!).

(b) 100% hydrogen (ionized)

Solution:

$$\frac{1}{\mu_I} = \frac{1}{1} = 1$$

$$\frac{1}{\mu_e} = \frac{1}{1} = 1$$

$$\frac{1}{\mu} = 1 + 1 = 2,$$

thus $\mu = 0.5$, $\mu_I = 1$, and $\mu_e = 1$. 

The percentages used to describe the composition in this problem are the mass fractions.
(c) 70% H / 30% He (fully ionized)

Solution:

\[
\begin{align*}
\frac{1}{\mu_I} &= (0.70) \frac{1}{1} + (0.30) \frac{1}{4} = \frac{31}{40} \\
\frac{1}{\mu_e} &= (0.70) \frac{1}{1} + (0.30) \frac{1}{2} = \frac{17}{20} \\
\frac{1}{\mu} &= \frac{13}{16} + \frac{7}{8} = \frac{13}{8},
\end{align*}
\]

thus \( \mu = \frac{8}{13} \approx 0.6, \mu_I = \frac{40}{31} \approx 1.3, \) and \( \mu_e = \frac{20}{17} \approx 1.2. \)

(d) 50% C / 50% O (fully ionized)

Solution:

\[
\begin{align*}
\frac{1}{\mu_I} &= (0.5) \frac{1}{12} + (0.5) \frac{1}{16} = \frac{7}{96} \\
\frac{1}{\mu_e} &= \frac{1}{2} \\
\frac{1}{\mu} &= \frac{7}{96} + \frac{1}{2} = \frac{55}{96},
\end{align*}
\]

thus \( \mu = \frac{96}{55} \approx 1.75, \mu_I = \frac{96}{7} \approx 13.7, \) and \( \mu_e = 2. \)

(e) Assume all these gases have the same \( \rho \) and \( T, \) which has the highest ideal gas pressure?

Solution: \( \mu \) is smallest for the 100% ionized H case, thus \( P \) is highest.
Energy Transport

2. Fick’s first law of diffusion is

\[ F_W = -D \frac{dw}{dx} \]

where \( w \) is the volume density of whatever (W) is diffusing, \( D \) is the diffusion coefficient, and \( F_W \) is the diffusive flux of \( W \).

(a) What must the dimensions of the diffusion coefficient be?

Solution:

\[ \frac{[W]}{[L]^2[T]} = [D] \frac{[W][L]^{-3}}{[L]} \implies [D] = \frac{[L]^2}{[T]} \]

(b) Construct a quantity with the same dimensions in terms of relevant microscopic quantities.

Solution: We want to think about physical quantities which could be important for particles diffusing. One is the distance particles travel between collisions, which is the mean free path. Another, is the speed at which the particles are traveling (\( v_{th} \)). From these two quantities, something with the correct dimensions is

\[ D \sim v_{th} \lambda. \]

(c) Write the equation for energy transport in a classical gas.

Solution: Plug the energy density in \( (E \sim nk_B T) \) to find the energy flux, so

\[ F_E \sim -(v_{th} \lambda)nk_B \frac{dT}{dx} = -k_B \frac{v_{th}}{c} \frac{dT}{dx} \]

(d) Write the equation for energy transport in a gas of photons.

Solution: Plug the energy density in \( (E \sim aT^4) \) to find the energy flux, so, noting that for photons \( v = c \),

\[ F_E \sim -(c\lambda)aT^3 \frac{dT}{dx} = -aT^3 \frac{c}{\kappa \rho} \frac{dT}{dx} \]
3. Fick’s second law of diffusion is

\[ \frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2} \]

where \( w \) and \( D \) have the same meaning as in the 1st law.

(a) This equation seems to imply that if you have a linear trend in the density \( w \) (\( \partial w/\partial x = \text{constant} \)), that \( w \) doesn’t evolve in time, even though Fick’s 1st law implies there’s a flux of \( W \). Why?

\textbf{Solution:} If you have a linear trend in \( w \), the flux into a region is equal to the flux out of a region, so the net change of \( w \) in that region is zero. What leads to a change in \( w \) is a difference the in/out fluxes, which is why Fick’s 2nd law has a second spatial derivative.

(b) Using dimensional analysis, estimate the time for ‘whatever’ to diffuse over a distance \( L \).

\textbf{Solution:} The diffusion coefficient has dimensions

\[ [D] = [L]^2[T]^{-1} \]

so

\[ t_{\text{diffusion}} \sim \frac{L^2}{D} \]

This also implies that the distance that something has diffused grows in time as \( \sqrt{t} \), which is a result that should be familiar if you’ve every studied random walks.

(c) Roughly, what is the timescale for thermal energy to diffuse across the classroom?

\textbf{Solution:} The diffusion coefficient is

\[ D \sim v\lambda \sim (300 \text{ m/s}) \frac{m}{\rho_\sigma} \sim \frac{(3 \times 10^4 \text{ cm/s})(5 \times 10^{-23} \text{ g})}{(10^{-3} \text{ g cm}^{-3})(10^{-15} \text{ cm}^2)} \]

so

\[ D \sim 1 \text{ cm}^2 \text{s}^{-1} \]

That means, for a room of size \( L \sim 10 \text{m} \)

\[ t_{\text{diffusion}} \sim 10^6 \text{ s} \]

(d) Are there any other transport processes that might also be occurring in this room?

\textbf{Solution:} Air currents in the room might be moving \( \sim 10 \text{ cm/s} \). Then, the time for this to transport something across the room would be \( \sim 100 \text{ s} \). These bulk motions would seem to be more effective than diffusion.
4. The thermal conductivity $K$ is defined such that

$$F_E = -K \frac{dT}{dx} .$$

(a) Using your result from the problem 2, write an expression for the thermal conductivity of an ideal gas.

**Solution:** Comparing the form of 2(c) with the equation for thermal conductivity, we see

$$K = k_B \frac{v_{th}}{\sigma} .$$

(b) Estimate the thermal conductivity of air.

**Solution:** Plugging in numbers,

$$K \sim (10^{-16} \text{ergs/K}) \frac{3 \times 10^4 \text{cm/s}}{10^{-15} \text{cm}^2} \sim 3 \times 10^3 \frac{\text{ergs}}{\text{cm s K}} .$$

(c) Estimate the thermal conductivity of the sun.

**Solution:** The velocity of electrons (which are what is doing the conduction) is

$$v_e \sim \sqrt{\frac{kT}{m}} \sim \sqrt{\frac{(10^{-16} \text{erg/K})(10^7 \text{K})}{10^{-27} \text{g}}} \sim 10^9 \text{cm/s} .$$

The coulomb cross section is

$$\sigma_c \sim \frac{e^4}{m_e^2 v_e^4} \ln \Lambda \sim \frac{(5 \times 10^{-10} \text{esu})^4}{(10^{-27} \text{g})^2(10^9 \text{cm/s})^4}(10) \sim 10^{-18} \text{cm}^2 .$$

So, then the thermal conductivity is

$$K \sim (10^{-16} \text{ergs/K}) \frac{10^9 \text{cm/s}}{10^{-18} \text{cm}^2} \sim 10^{11} \frac{\text{ergs}}{\text{cm s K}} .$$

If you get a chance, use this conductivity to check that conduction can’t transport the bulk of the energy in the sun.