Temperature & Nuclear Fusion

4 October 2011

Goals

• Review temperature in stars
• Practice using the important energy scales for nuclear fusion

Temperature

1. For each relation we regularly use in class, determine whether it is appropriate to use $T_c$ or $T_{\text{eff}}$ and explain why.

(a) In the Virial Theorem:

$$kT \sim \frac{GMm_p}{R}.$$  

**Solution:** The Virial Theorem relates the total potential energy to the total kinetic energy. That means the $T$ here is an average temperature inside the star; most of the star is at a temperature closer to the central temperature than the surface temperature, so use the central temperature.

(b) In the radiative diffusion equation:

$$F = -\frac{4c}{3} \frac{d(aT^4)}{dr}.$$  

**Solution:** The radiative diffusion equation makes a local statement about the flux, so the $T$ that appears is the local temperature. When making our estimates for the sun, we made the approximation $dT/dr \sim T_c/R_\odot$, so we frequently used the core temperature in this equation.

(c) In the Stefan-Boltzmann equation:

$$L = 4\pi R^2 \sigma T^4.$$  

**Solution:** This $T$ is the effective temperature, as we are looking at the surface of the star ($4\pi R^2$ is the surface area of the star).

(d) In fusion energy generation:

$$\epsilon \propto \rho^\alpha T^\beta.$$  

**Solution:** Fusion occurs in the core, so naturally the important temperature is the central temperature.
Nuclear Fusion

2. There are a few different energy scales in fusion, which we will review here.

(a) What is the thermal energy, and how is it important to fusion?

Solution: This is just \( \sim kT \), and sets the typical classical distance of closest approach for two particles.

(b) What is the Gamow energy, and how is it important to fusion? How does it depend on temperature?

Solution: The Gamow energy is the energy at which the probability of a particle tunneling through the Coulomb barrier to fuse with another particle is significant (\( e^{-1} \)). It is temperature independent. The relevant variables are the charges of the fusing nuclei and the reduced mass.

(c) How do these energies relate to the typical energy at which fusion reactions occur?

Solution: The Gamow energy is way out on the tail of the Maxwell-Boltzmann distribution, so there are very (!) few particles able to tunnel with high probability. Nuclear reactions then typically occur at an energy such that the tunneling probability is not too low, and also is not too far out on the MB tail so that there are no particles at that energy. We found in class that this energy is

\[
E_0 = \left( \frac{1}{2} E_G^{1/2} kT \right)^{2/3}.
\]

3. Fusion of heavier elements requires higher temperatures - you can tell this from \( E_G \), which increases for higher \( Z \) elements. How does a star achieve these higher temperatures?

Solution: Recall the expression for temperature via the virial theorem:

\[
kT = \frac{GM\mu m_p}{3R}.
\]

Since mass is mostly conserved, the main way to achieve higher temperatures is to contract, so that \( R \) decreases. Another more subtle way to increase temperature is by having a higher proportion of heavier elements in the star, such that \( \mu \) increases.
4. Written in terms of energy, the Maxwell-Boltzmann distribution is:

\[
f(E)dE = \frac{2}{kT} \left( \frac{E}{\pi kT} \right)^{1/2} e^{-E/kT} dE.
\]

Calculate the following for a particle in the center of the sun.

(a) What’s the probability of having an energy \( E \sim kT \)?

\[
p(E \sim kT) \approx f(kT)kT \approx \frac{2}{\sqrt{\pi}} e^{-1} \approx 0.4
\]

(b) What’s the probability of having an energy \( E \sim E_G \)?

\[
p(E \sim E_G) \approx f(E_G)E_G \approx 500 \sqrt{\frac{250}{\pi}} e^{-250} \approx 10^{-105}
\]