Nuclear Reactions & Scaling Arguments
11 October 2011

Goals
- Review nuclear reaction rates
- Practice using scaling arguments

Nuclear Reactions

1. Consider the simple reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$. The $k_i$ are the rate constants for each step. A larger $k$ means a faster reaction.

(a) Describe what happens when $k_1 \gg k_2$. What is the effective rate constant for the conversion of $A$ to $C$?

Solution: $A \rightarrow B$ goes very quickly, so immediately all of $A$ is converted into $B$. Then, $B \rightarrow C$ at a much slower rate. So the total rate can just be effectively described by the rate constant of the rate-limiting step, $k_2$.

(b) Describe what happens when $k_1 \ll k_2$. What is the effective rate constant for the conversion of $A$ to $C$?

Solution: $B \rightarrow C$ goes very quickly, so as soon as some $B$ appears is converted into $C$. However, $A \rightarrow B$ at a much slower rate. So the total rate can just be effectively described by the rate constant of the rate-limiting step, $k_1$.

2. Why do reaction rates depend on density in the way that they do?

(a) Explain in words and/or pictures why the reaction rate for something like $p + p \rightarrow D$ scales like $n_p^2$.

Solution: One heuristic way is to think about the number density as being proportional to the probability of a single particle being at a location. If you want to find the probability of having two particles in the same spot, that probability is the product of the individual particle probabilities. So for two particles that goes like $n^2$.

(b) Suppose that the reaction $4 \times ^1H \rightarrow ^4He$ happened by the simultaneous collision of 4 protons. How would $\epsilon$ scale with $\rho$?

Solution: This would be a four-body reaction, so the rate would go like $n^4$. However, $\epsilon$ is the rate per unit mass, so it will scale like one power lower, meaning $a = 3$. 

Think in microscopic terms.
3. Frequently, we approximate nuclear reaction rates with the simple form \( \varepsilon \propto \rho^\alpha T^\beta \). On this plot of \( \varepsilon \) vs. \( T \) (at fixed \( \rho \)), do the following:

(a) Mark the central temperature of the sun

Solution: \( T_{c,\odot} \approx 1.5 \times 10^7 \) K

(b) Explain the meaning of \( \beta \) by sketching on the graph

Solution: On a log-log plot, \( \beta \) is the slope of the tangent line at the temperature of interest.

(c) Find an approximate value of \( \beta \) (at \( T_{c,\odot} \))

Solution: You should find \( \beta \approx 4.68 \).
Scaling Arguments

4. In lecture, Eliot mentioned that fusion in the solar corona is negligible. We want to confirm his statement and practice exploiting the convenience of scaling arguments. This plot gives us a rough structure of the corona.

![Plot showing the temperature and density distribution in the solar atmosphere.

(a) What is the amount of mass in the solar corona? Approximate it as a spherical shell of constant density.

**Solution:** The number density is $n \sim 3 \times 10^8$ cm$^{-3}$, so $\rho \sim 5 \times 10^{-16}$ g cm$^{-3}$. The thickness looks to be something like $\Delta R \sim 10^{10}$ cm. The radius of the shell is roughly the solar radius, $R_\odot \sim 7 \times 10^{10}$ cm. So then

$$M \sim 4\pi R_\odot^2 \Delta R \rho \sim 2.5 \times 10^{17} \text{ g} \sim 10^{-16} M_\odot$$

(b) At fixed temperature, write down a proportionality for the hydrogen fusion luminosity in terms of $\rho$ and $M$.

**Solution:**

$$L \propto \epsilon M \propto \rho M$$

Relating $\rho$ to $M$ assumes something about the geometry. We’re going to want to apply our scaling to two different geometries, sphere and shell, so it’s best to leave it as is.

(c) Estimate the luminosity (from fusion) of the solar corona.

You should get a rough density, radius and thickness from the plot.

Don’t try and relate $\rho$ to $M$. In doing so, what would you be assuming?

Use what you know about the sun to anchor a scaling relation.
Solution: We roughly know the values of $L$, $\rho$ and $M$ appropriate for the sun. So we can anchor our scaling to the sun by writing

$$L \sim L_\odot \left( \frac{\rho}{100 \text{ g cm}^{-3}} \right) \left( \frac{M}{0.1 M_\odot} \right).$$

Plugging in the values we found in (a), we get

$$L_{\text{corona,nuc}} \sim 10^{-33} L_\odot.$$