1. **The Helium Main Sequence**

In certain (later) stages of stellar evolution, stars are largely composed of He and He fusion dominates the stellar luminosity. One can approximate such stars as lying on a He main sequence. In the first part of this problem we will calculate the properties of the He main sequence assuming that a star is composed of pure He, that energy transport is via radiation, that electron scattering dominates the opacity, and that gas pressure dominates. Note that throughout this problem you should not just give scaling laws for the desired relations; you should also determine reasonable normalizations. In the last part of the problem we will do some MESA calculations that include more realistic models with a He core and a H envelope.

(a) Calculate the mass-luminosity relationship for the He main sequence.

(b) Estimate the core temperature of a 1 solar mass He star. You do not need to do the full integral $L_{\text{fusion}} = \int dM \epsilon$, but can approximate this as $L_{\text{fusion}} \sim 0.1 M \epsilon(r = 0)$. Given your result for $T_c$ for a 1 $M_{\odot}$ star, calculate the power-law relation $T_c(M)$.

(c) Use your results above to determine the $R(M)$ and $T_{\text{eff}}(L)$ relations for the He main sequence. Then sketch the relative positions of the H & He main sequences in the HR diagram.

(d) What is the He main sequence lifetime as a function of stellar mass? Compare this to the corresponding H burning lifetime.

(e) We associate the “horizontal branch” in the HR diagram with stars burning helium in their cores. But in part (c), the He main sequence you found wasn’t horizontal.

In the MESA portion of this problem, we will study more realistic horizontal branch stars. It turns out that it is easier to do so by evolving a stellar model in an “unphysical” way. Instead of the MESA result representing the evolutionary path of a single star over time, the result at each timestep will represent a different star. We will provide you with an initial model of a 0.35 $M_{\odot}$ pure helium-burning star. In MESA, we will turn off the change of composition due to nuclear burning, but keep the energy generation rate the same. This makes the He main sequence lifetime infinite. Then we will have the model accrete material on a timescale much longer than the thermal timescale, so that the model comes into thermal equilibrium just as a real star of that mass and composition would be. Now, at each timestep the model will represent a star of a different mass on the helium main sequence.

(i) Accrete helium onto the initial model until the total mass reaches 2.0 $M_{\odot}$. Plot $R(M)$ and $T_{\text{eff}}(L)$ from the MESA evolution and compare with your previous analytic results.

(ii) Now explore the effect of adding a hydrogen envelope. Run the model as in part (i), but instead of stopping at 2.0 $M_{\odot}$, stop at the mass you were assigned via an email from Josiah. Now, change from accreting pure helium, to a mixture that is mostly hydrogen. (See the email from Josiah for instructions on how to do this.) Continue accreting up until the mass of the hydrogen envelope is equal to the mass of the core. This evolution represents a sequence of “He MS” stars with a H envelope mass ranging from zero up to the mass of the He core.

Send the output of your He + H MESA models to Josiah by Wed, April 24. Then answer the following questions:

(iii) Give two reasons why the presence of a hydrogen envelope affects the location of the star in the HR diagram.

(iv) Based on the results from all of the class calculations, in order to produce a truly horizontal branch on the HR diagram, does the ratio of the envelope mass to the core mass need to increase, decrease, or remain constant as the core He mass increases?
2. The Mass-Radius Relation for White Dwarfs

Equation 3.53 of HKT (3.50 of HK) gives the general expression for the pressure of a cold degenerate gas. In the extreme non-relativistic or relativistic limits, we can write the degenerate equation of state in polytropic form, but this is not possible in general.

Write a program to solve the equations of hydrostatic equilibrium and \( \frac{dM_e}{dr} = 4\pi r^2 \rho \) using the general equation of state of a cold degenerate electron gas. This solution provides the density and pressure profiles and the mass-radius relationship for such objects. Boundary conditions on these differential equations only need to be provided at the center of the star so you can integrate from the center to the surface just like you did for polytropes in HW 1. You might find it easiest to write \( \frac{dP}{dr} = \left( \frac{dP}{dp} \right) \left( \frac{dp}{dr} \right) \) and solve for \( \frac{dP}{dp} \) from the EOS. Since \( \frac{dP}{dp} \) is just a function of \( \rho \) (as you can show from the EOS), this technique eliminates \( P \) as an explicit variable that you have to worry about.

a) By varying the central density you get a one-parameter sequence of stars with different masses \( M \) and radii \( R \). Plot the resulting mass-radius relationship for \( \mu_e = 2 \) (a C/O WD) and \( \mu_e = 56/26 = 2.15 \) (an iron WD). Explain the behavior of your numerical results as \( R \to 0 \) and \( R \to \infty \) in terms of the polytropic models discussed in class. In addition, for a 0.6\( M_\odot \) and 1\( M_\odot \) \( \mu_e = 2 \) WD, compare your numerical result to the \( \gamma = 5/3 \) polytropic result for \( R(M) \) derived in class.

b) The results you have derived in a) should show that as \( M \to 0, R \to \infty \). As we discussed in our lecture on brown dwarfs, however, this is not correct because Coulomb interactions become important in the equation of state of low-mass objects (brown dwarfs and planets). Estimate the density at which the Coulomb energy per particle becomes comparable to the Fermi energy. What mass and radius does this correspond to? This is a very rough estimate of the maximum radius of a degenerate object.

c. Type 1a supernovae are believed to occur when carbon fusion begins in a WD under highly degenerate conditions and leads to the explosion of the star. This can occur if the mass of the WD approaches the Chandrasekhar mass (probably because of accretion, though models in which mergers of two WDs do the job have also been presented). As your calculations in a) show, as \( M \to M_{ch} \), the central density of the WD increases. The central temperature does as well because of compressional heating. At masses sufficiently close to \( M_{ch} \), carbon fusion begins when energy generation by fusion exceeds energy losses by neutrinos. The star explodes soon thereafter. Detailed models predict that carbon fusion begins when \( \rho_c \approx 3 \times 10^9 \) g cm\(^{-3} \) (and \( T_c \approx 7 \times 10^8 \) K, but the temperature is not important for this problem).

i) Use your results from a) to determine the mass and radius of the WD when it explodes. Assume \( \mu_e = 2 \).

ii) What is the net binding energy \( E_{tot} \) of the WD at this time (don’t worry about the thermal energy, which is negligible)? Recall that when \( \gamma \to 4/3 \) \( E_{tot} \to 0 \). To quantify just how close the WD is to this limit, it is useful to compare the WD’s binding energy to its gravitational potential energy.

iii) If the entire star undergoes carbon fusion, show that sufficient energy is released to unbind the entire star. Estimate the asymptotic kinetic energy and velocity of the ejecta.

---

\(^1\)Chandrasekhar Nobel Prize (1983)
3. **Zero Metallicity Stars**

Consider a primordial massive star \((M \sim 100M_{\odot})\) formed at high redshift. It initially has no metals but has both hydrogen \((X \approx 0.75)\) and helium \((Y \approx 0.25)\). Recall that massive stars such as this one are supported by radiation pressure.

a) Show that He fusion, not H fusion via the pp chain, is somewhat more important for energy generation in such stars. Hint: Calculate the temperature at which each fusion process would produce the luminosity of the star, making reasonable assumptions about the radius/density.

b) Now estimate the \(L(M), T_c(M), R(M),\) and \(T_{eff}(L)\) relations for zero metallicity massive stars. As in problem 1, determine appropriate normalizations for all of your scaling relations.

c) Carbon and Oxygen are products of He fusion. Thus once He fusion has occurred for some amount of time, there will be enough C & O around for H fusion to proceed by the CNO cycle.

i) Estimate the CNO mass fraction \(Z_{CNO}\) required for CNO fusion to dominate over He fusion in the centers of primordial stars.

ii) Roughly how long does it take the star to generate this amount of CNO?

Rather counter-intuitively, what this means is that primordial massive stars spend the vast majority of their lives fusing \(H \rightarrow He\) via the CNO cycle, using the trace amount of CNO generated by a brief epoch of He fusion!

iii) When fusion proceeds via the CNO cycle instead of He fusion, will your results for part b) change significantly? Why or why not? You don’t need to give a quantitative answer for this part of the problem, but just explain your answer physically.

d) Use the provided MESA inlists to evolve a \(100M_{\odot}\) star with \(Z = 0\) on to the ZAMS.

Plot the energy generation from each of the three processes of interest: pp, CNO, and \(3\alpha\) as a function of time. Also plot the central C,N,O abundances. Compare the results with your estimates in the previous parts of the problem.

When you compare your MESA results to the calculation above, you will find that the same qualitative picture holds (CNO dominates the energy generation rate), but the quantitative details differ in interesting ways. This is because the triple-alpha energy generation ends up not dominating the pp energy generation. Our estimate in part (a) is incorrect for two reasons: (1) the usual expression for the pp-rate underestimates the true rate by a factor of 3 at these high temperatures and helium-rich conditions; and (2) the triple-alpha rate is more temperature and density sensitive and so a smaller portion of the mass of the star participates in helium fusion than in pp-fusion. Your estimate in a) above [probably] does not take this into account, though it could be modified to do so.