Cosmological Perturbation Theory

Jordan Carlson

Lawrence Berkeley National Laboratory

June 6, 2008
Outline

1 Introduction
   - Model and assumptions
   - Fluid equations

2 Perturbation Theory
   - Linear theory
   - Standard Perturbation Theory

3 Alternatives and extensions to SPT
Goal: understand large-scale structure

- Baryon acoustic oscillations imprint characteristic scale on matter distribution (Standard Ruler)
- Matter fluctuations amplified by growth function
- Dark energy dominates growth function today (at low $z$)
- Therefore measuring matter distribution today tells us about dark energy!
Why perturbation theory

- Need to run large number of N-body simulations to compute statistical observables (e.g. power spectrum)
- BAO scale is large ($\sim 100 \text{ Mpc/h}$), so need to run large volume simulations
- Simulations are expensive!
- Analytic solution computes statistical quantities directly
- Direct analytic solution impossible (non-linear equations of motion), so must resort to perturbation theory
Start well after matter-radiation equality

Flat FRW cosmology with $\Lambda$, ignore radiation and neutrinos

Friedmann equation: $H^2 = \frac{8\pi G}{3} \bar{\rho} + \frac{\Lambda}{3}$

Mean density: $\bar{\rho} \propto a^{-3}$

Later will restrict attention to Einstein-de Sitter cosmology: $\Omega_m = 1, \Lambda = 0$
Matter fluid

- Newtonian gravity (distance scales well within the horizon)
- Non-relativistic fluid
- Pressureless, collisionless, zero viscosity
- Assumptions good for cold dark matter
- Assumptions fail for baryons, but only in regions of high density
Peculiar velocity field

- Single-stream approximation (no shell crossing)
- Irrotational: vorticity $\mathbf{w} \equiv \nabla \times \mathbf{v} = 0$
- True in linear theory: $\mathbf{w}$ decays as $a^{-1}$
- (Not clear when/where these assumptions break down, but the show must go on)
Cosmological coordinates

- Comoving coordinates: \( x = r / a \)
- Conformal time: \( \tau = \int dt / a \) or \( d\tau = dt / a \)
- Metric: \( ds^2 = a^2(\tau)[-d\tau^2 + dx^2] \)
Equations of motion for a single particle

- Non-relativistic action:

\[ S = \int dt \left[ \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 - m\Phi \right] \]

\[ = \int d\tau a(\tau) \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 - m\Phi \right] \]

\[ \Phi(x, \tau) = a^2(\tau) \int d^3x' \frac{\delta\rho(x', \tau)}{|x - x'|} \]

- Equations of motion:

\[ \frac{dx}{d\tau} = \frac{p}{ma}, \quad \frac{dp}{d\tau} = -ma \nabla \Phi \]
Phase space distribution function

\[ dN = f(x, p, \tau) \, d^3x \, d^3p \]

For a collection of point masses,

\[ f(x, p, \tau) = \sum_\alpha \delta^3(x - x_\alpha(\tau)) \, \delta^3(p - p_\alpha(\tau)) \]

- Mass density: \( \rho(x, \tau) = ma^{-3}(\tau) \int f(x, p, \tau) \, d^3p \)
- Momentum density:
  \[ \rho(x, \tau)v(x, \tau) = a^{-4}(\tau) \int f(x, p, \tau) \, p \, d^3p \]
- All higher moments of \( f \) are products of \( \rho \) and \( v \)
Collisionless Boltzmann equation

\[
\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{p}{ma} \cdot \nabla f - ma\nabla \Phi \cdot \frac{\partial f}{\partial p} = 0
\]

- Conservation of phase space volume
- Taking moments gives fluid equations
Fluid equations

\[ \frac{\partial \delta}{\partial \tau} + \nabla \cdot \mathbf{v} = -\nabla \cdot (\delta \mathbf{v}) \]  
(Continuity)

\[ \frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H} \mathbf{v} + \nabla \Phi = -(\mathbf{v} \cdot \nabla) \mathbf{v} \]  
(Euler)

\[ \nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \delta = \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) \delta \]  
(Poisson)

- \( \mathcal{H} = \frac{d \ln a}{d \tau} = a \dot{H} \)
- \( \rho(\mathbf{x}, \tau) = \bar{\rho}(\tau) [1 + \delta(\mathbf{x}, \tau)] \)
- \( \mathbf{v} = \) peculiar velocity \( (\mathbf{v} = 0 \) at zeroth order)
Assume $\delta$ and $v$ small, of the same order

Drop right-hand sides of fluid equations:

\[
\begin{align*}
\frac{\partial \delta}{\partial \tau} + \theta &= 0 \\
\frac{\partial \theta}{\partial \tau} + \mathcal{H} \theta + \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2 \delta &= 0
\end{align*}
\]

$\Rightarrow$

\[
\frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta}{\partial \tau} - \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2 \delta = 0
\]

$\theta \equiv \nabla \cdot v$ (peculiar velocity divergence)
Growth function

\[
\frac{d^2 D}{d\tau^2} + \mathcal{H}(\tau) \frac{dD}{d\tau} - \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) D = 0
\]

- Two linearly independent solutions: \( D_+(\tau) \) (growing) and \( D_- (\tau) \) (decaying)
- Ignore decaying solution: \( \delta_L(x, \tau) = D_+(\tau) \delta_0(x) \)
- For Einstein-de Sitter universe (or during matter domination), \( D_+ \propto a \) and \( D_- \propto a^{-3/2} \)
- When \( \Lambda \neq 0 \), \( D_+ \) falls below \( a \) at late times
Introduction

Perturbation Theory

Alternatives and extensions to SPT

Linear theory

Standard Perturbation Theory

Linear growth function plot

Linear growth function for $\Lambda \neq 0$

$D_+(a)$

$0 \ 0.1 \ 0.2 \ 0.3 \ 0.4 \ 0.5 \ 0.6 \ 0.7 \ 0.8 \ 0.9 \ 1$

$0 \ 0.1 \ 0.2 \ 0.3 \ 0.4 \ 0.5 \ 0.6 \ 0.7 \ 0.8 \ 0.9 \ 1$

Jordan Carlson

Cosmological Perturbation Theory
Statistical observables

- Correlation function: $\langle \delta(x)\delta(x') \rangle = \xi(|x - x'|)$
- Baryon acoustic peak at $r \approx 105$ Mpc/h
- Power spectrum: $\langle \tilde{\delta}(k)\tilde{\delta}(k') \rangle = P(k)\delta^3(k + k')$
- $P(k)$ is just the Fourier transform of $\xi(r)$
- At linear order $P_L(k, \tau) = D^2(\tau)P_0(k)$
Correlation function

Linear correlation function at $z = 0$
Power spectrum

Linear power spectrum at $z = 0$

$$P_L(k) \left[ (\text{Mpc}/h)^3 \right]$$

$k \left[ h/\text{Mpc} \right]$
**Introduction**

Perturbation Theory

Alternatives and extensions to SPT

Linear theory

Standard Perturbation Theory

Power spectrum

Linear power spectrum at $z = 0$ divided by no-wiggle form
Basic theory worked out long ago [reviewed in Peebles 1980]
Explicit formulas and diagrammatic methods developed in 80’s and 90’s [Fry 1984, Goroff et al 1986, Makino et al 1992]
Basis for most other perturbative theories
Fluid equations in Fourier space

- Velocity field: \( \tilde{v}(k) = -\frac{i k}{k^2} \tilde{\theta}(k) \)
- RHS of continuity equation:
  \[
  \text{FT}[-\nabla \cdot (\delta \mathbf{v})] = -i k \cdot \int d^3q_1 \, d^3q_2 \, \delta^3(q_1 + q_2 - k) \frac{-iq_1}{q_1^2} \tilde{\theta}(q_1) \tilde{\delta}(q_2)
  \]
- RHS of Euler equation (after taking divergence):
  \[
  \text{FT}[-\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}]] = -i k \cdot \int d^3q_1 \, d^3q_2 \, \delta^3(q_1 + q_2 - k) \times \left( \frac{-iq_1}{q_1^2} \cdot iq_2 \right) \frac{-iq_2}{q_2^2} \tilde{\theta}(q_1) \tilde{\theta}(q_2)
  \]
Fluid equations in Fourier space

\[
\frac{\partial \tilde{\delta}}{\partial \tau} + \tilde{\theta} = - \int d^3q_1 \, d^3q_2 \, \delta^3(q_1 + q_2 - k) \frac{k \cdot q_1}{q_1^2} \tilde{\theta}(q_1) \tilde{\delta}(q_2),
\]

\[
\frac{\partial \tilde{\theta}}{\partial \tau} + \mathcal{H} \tilde{\theta} + \frac{3}{2} \Omega_m \mathcal{H}^2 \tilde{\delta}
\]

\[
= - \int d^3q_1 \, d^3q_2 \, \delta^3(q_1 + q_2 - k) \frac{k^2(q_1 \cdot q_2)}{2q_1^2q_2^2} \tilde{\theta}(q_1) \tilde{\theta}(q_2).
\]

- Non-linearity manifested as convolution in Fourier space (mode-coupling)
Perturbation expansion

\[ \tilde{\delta}(k, \tau) = \sum_{n=1}^{\infty} \tilde{\delta}^{(n)}(k, \tau), \quad \tilde{\theta}(k, \tau) = \sum_{n=1}^{\infty} \tilde{\theta}^{(n)}(k, \tau). \]

- Insert perturbation expansion in fluid equations, solve order by order
- Simplification for Einstein-de Sitter universe:

\[ \tilde{\delta}^{(n)}(k, \tau) = a^n(\tau) \delta_n(k), \quad \tilde{\theta}^{(n)}(k, \tau) = \mathcal{H}(\tau) a^n(\tau) \theta_n(k) \]

- \( a \propto \tau^2, \mathcal{H} = 2/\tau \)
Recursive solution

\[ n \delta_n(k) + \theta_n(k) = A_n(k), \quad 3\delta_n(k) + (1 + 2n)\theta_n(k) = B_n(k), \]

where

\[
A_n(k) = -\int d^3q_1 d^3q_2 \delta^3(q_1 + q_2 - k) \frac{k \cdot q_1}{q_1^2} \sum_{m=1}^{n-1} \theta_m(q_1) \delta_{n-m}(q_2),
\]

\[
B_n(k) = -\int d^3q_1 d^3q_2 \delta^3(q_1 + q_2 - k) \frac{k^2(q_1 \cdot q_2)}{2q_1^2 q_2^2} \times \sum_{m=1}^{n-1} \theta_m(q_1) \theta_{n-m}(q_2).
\]

- Plug in to fluid equations: \( n \)th order term sourced by lower orders
Integral solution

- Can obtain explicit integral expression

\[
\delta_n(k) = \int d^3q_1 \ldots d^3q_n \delta^3(\sum q_i - k) F_n(\{q_i\}) \delta_1(q_1) \ldots \delta_1(q_n)
\]

\[
\theta_n(k) = \int d^3q_1 \ldots d^3q_n \delta^3(\sum q_i - k) G_n(\{q_i\}) \delta_1(q_1) \ldots \delta_1(q_n)
\]

- Kernels \(F_n, G_n\) defined recursively, first few are

\[
F_1(q_1) = G_1 = 1
\]

\[
F_2(q_1, q_2) = \frac{5}{7} + \frac{q_1 \cdot q_2}{2q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \left( \frac{q_1 \cdot q_2}{q_1 q_2} \right)^2
\]

\[
G_2(q_1, q_2) = \frac{3}{7} + \frac{q_1 \cdot q_2}{2q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{4}{7} \left( \frac{q_1 \cdot q_2}{q_1 q_2} \right)^2
\]
Assume initial density $\delta_0$ is a Gaussian random field, so all $n$-point functions reduce to products of 2-point function. Expand $\delta$ to third order to obtain $P(k)$ to second order:

$$\langle \tilde{\delta}(k)\tilde{\delta}(k')\rangle = a^2\langle \tilde{\delta}_1(k)\tilde{\delta}_1(k')\rangle + a^4\langle \tilde{\delta}_2(k)\tilde{\delta}_2(k')\rangle + a^4\langle \tilde{\delta}_1(k)\tilde{\delta}_3(k')\rangle + a^4\langle \tilde{\delta}_3(k)\tilde{\delta}_1(k')\rangle$$

$$\implies P_2(k) = P_L(k) + P_{22}(k) + P_{13}(k)$$

Explicit integral expressions exist for $P_{22}$ and $P_{13}$. Schematically $P_{22} \sim \int P_L \int P_L$, $P_{13} \sim P_L \int P_L$
Limitations of SPT

- Only formally valid for Einstein-de Sitter universe: $D \propto a$
  - Approximately valid for arbitrary cosmology if we just replace $a$ by the true linear growth function $D$ in our perturbation expansion
- Perturbation theory breaks down at late times or at high $k$ ($\sim k = 0.2h$/Mpc at $z = 0$)
- Power spectrum diverges, can’t calculate correlation function meaningfully
Growth of non-linear power

$P(k) \,[\,\text{Mpc}/h]^3$

$z = 19.00$

$k \,[\,\text{h/Mpc}]$

Linear

2nd order

Jordan Carlson
Cosmological Perturbation Theory
Growth of non-linear power

\[ P(k) / P_{nw}(k) \]

- Linear
- 2nd order

\( z = 19.00 \)

\( k [h/\text{Mpc}] \)
Comparison with N-body simulations

The graph shows a comparison between the linear theory, second order SPT, and N-body data. The $P(k)/P_{nw}(k)$ ratio is plotted against $k$ [h/Mpc]. The N-body data points are represented by blue circles, while the red line and green line indicate the linear theory and second order SPT, respectively.
Lagrangian Perturbation Theory

- Lagrangian picture of fluid mechanics: $x = q + \Psi(q)$
- $\bar{\rho}[1+\delta(x)]d^3x = \rho d^3q \implies 1+\delta(x) = [\det(\delta_{ij} + \partial \Psi_i/\partial q_j)]^{-1}$
- Linear solution for $\Psi$ gives Zel’dovich approximation:
  \[
  1 + \delta(x, \tau) = \frac{1}{[1 - \lambda_1 D_1(\tau)][1 - \lambda_2 D_1(\tau)][1 - \lambda_3 D_1(\tau)]}
  \]
- Pros: intrinsically non-linear, 2nd and 3rd order calculations feasible
- Cons: breaks down at lower $k$ than SPT
Renormalized Perturbation Theory

- Crocce & Scoccimarro, astro-ph/0509418
- Starts with diagrammatic formulation of perturbation expansion
- Attempts to identify renormalized vertices and propagators, à la QFT
- Pros: seems to match simulation data well
- Cons: extremely complicated, requires field theory background
Renormalized Perturbation Theory
Matsubara,
\textit{arXiv:0711.2521}

Pulls out certain series of terms from infinite PT expansion, resums them into a Gaussian prefactor:
\[ P \sim e^{-A k^2} \left[ P_L(k) + \tilde{P}_{13}(k) + P_{22}(k) \right] \]

Power spectrum is wrong at high \( k \), but correlation function is good
Renormalization group techniques

- McDonald, astro-ph/0606028
- Macarrese and Pietroni, astro-ph/0703563
- Pietroni, arXiv:0806.0971
The future?

- Upcoming surveys need to be compared against accurate theoretical predictions to learn about dark energy
- Renewed interest in cosmological perturbation theory on many fronts
- Many new papers, with new techniques, appearing in recent years (even days!)
Bibliography

- M. Pietroni, 0806.0971.