The Phase Shift

Imagine a particle of mass m and energy E incident on a square potential

$$V(x) = \begin{cases} V_0 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with $V_0 < E$. Label the region $x < -a$ as I, $-a < x < a$ as II and $x > a$ as III. Since $V(x)$ is even we can decompose the solutions into even and odd parity modes which in region II are

$$\psi_{II,e} = \cos \kappa x \quad \text{and} \quad \psi_{II,o} = \sin \kappa x \tag{2}$$

where $\kappa = \sqrt{2m(E - V_0)}$ and we have set $\hbar = 1$ for simplicity. For now we consider only the even mode, the argument for the odd mode is completely analogous.

In regions I and III, $\psi_{II,e}$ must match onto even functions oscillating with period $2\pi/k$ where $k = \sqrt{2mE}$ so

$$\psi_{I,e} = A\cos(kx - \delta_e) \tag{3}$$

$$\psi_{III,e} = A\cos(kx + \delta_e) \tag{4}$$

where A and δ_e are constants. This satisfies the wave equation in all 3 regions and has the right parity. The only thing left is to match the function and its first derivative at $x = \pm a$. At $x = a$ we have

$$\cos \kappa a = A\cos(ka + \delta_e) \tag{5}$$

$$-\kappa \sin \kappa a = -kA\sin(ka + \delta_e) \tag{6}$$

from which we can solve for the phase shift as

$$\tan(ka + \delta_e) = \frac{\kappa}{k} \tan \kappa a \tag{7}$$

and then use Eq. (5) to solve for A. If we were to match at $x = -a$ we would obtain the identical equations.

An argument completely analogous to the above for the odd modes gives

$$\tan(ka + \delta_o) = \frac{k}{\kappa} \tan \kappa a \tag{8}$$