Sturm-Liouville Theory

(Martin White)

There is a large theory associated with the eigenvalues of certain classes of operator. This theory has several nice features which are useful in quantum mechanics, so we will review some of the basics here. The reader is referred to textbooks on mathematical methods for more information.

Consider the eigenvalue problem involving a Hermitean operator, the so-called *Sturm-Liouville differential equation*

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] - q(x)u(x) + \lambda\rho(x)u(x) = 0$$

for real $p(x)$, $q(x)$ and $\rho(x)$ with $\rho(x)$ non-negative on the interval in question. The operator $L = p\partial_x^2 + (\partial_x p)\partial_x - q$ is Hermitean so we can find solutions such that

$$u_i \cdot u_j \equiv \int_a^b \rho(x)\, dx\ u_i^*(x)u_j(x) = \delta_{ij}$$

The set of u_i can be shown to be a complete set under very general conditions.

The Sturm-Liouville equation is the Euler-Lagrange equation (that is to say it specifies the solution to $\delta I = 0$) for the functional

$$I\left[u(x)\right] = \int_a^b dx\ \ \left[p(u')^2 + (q - \lambda\rho)u^2\right]$$

or, more commonly, the extremum of

$$I\left[u(x)\right] = \int_a^b dx\ \ \left[p(u')^2 + qu^2\right]$$

with

$$J\left[u(x)\right] = \int_a^b \rho\, dx\ u^2 = \text{constant}$$

where λ is the Lagrange multiplier enforcing the J constraint. Note that I looks like a kinetic plus a potential energy term while J is a normalization condition.

It is possible to show that the series

$$\sum_i c_i u_i(x) \qquad \text{with } c_i = \int_a^b \rho\, dx f u_i$$

converges in the mean to $f(x)$. For all physical purposes this is just as good as ordinary (point-by-point) convergence.

Solutions to Sturm-Liouville problems frequently have very nice properties: recurrence relations relating the functions of different orders, and the functions and their derivatives, generating functions, generalized Rodrigues' formulae etc. See a text on mathematical methods for more information.