AY 202 Assignment 5

due: Tuesday, April 5

Problem 1: A rotating fluid tends to develop coherent substructures along the
direction of the rotation axis. Such *Taylor columns* are easily produced in the laboratory.
Let us explore the basis of the phenomenon.

(a) Consider motions within an incompressible, inviscid, rotating fluid. View the fluid
in a reference frame rotating with the angular velocity $\Omega$. If $u$ is the fluid velocity
in this frame, write out the conditions of mass and momentum conservation. For the
latter, let $\rho, p$, and $\Phi$ be the density, pressure, and gravitational potential, respectively.

(b) Define the reduced pressure $p'$ as

$$p' \equiv p + \rho \Phi - \frac{1}{2} \rho |\Omega \times r|^2.$$

Using $p'$ instead of $p$, derive a simplified version of the momentum equation.

(c) Now consider steady-state motion, slow enough that you can neglect the convective
portion of the acceleration $d\mathbf{u}/dt$. What is the momentum equation now?

(d) Take the curl of this last equation and utilize mass conservation to derive an
expression for the change in $\mathbf{u}$ in the direction of $\Omega$. This is the Taylor-Proudman
theorem.

Problem 2: Within the inertial range of a fully turbulent fluid, two particles start at
a distance $\lambda_1 \gg \lambda_0$, where $\lambda_0$ is the viscous dissipation scale. At what time $t$ are the
particles separated by a distance $\lambda_2$, where $\lambda_1 \ll \lambda_2 \ll L$? Here, $L$ is the outer scale
of the bulk flow. Solve this problem two ways:

(a) Use dimensional analysis to estimate $d\lambda/dt$ in terms of $\lambda$ itself and $\dot{\epsilon}$. Here, $\dot{\epsilon}$ is the
(invariant) energy transfer rate per unit mass per unit time.

(b) At each scale $\lambda$, there is an effective viscosity $\nu_{\text{turb},\lambda}$:

$$\nu_{\text{turb},\lambda} \equiv u_\lambda \lambda,$$

where $u_\lambda$ is the appropriate eddy velocity. Recall that the ordinary kinematic viscosity $\nu$
is also a diffusion coefficient. Regarding the separation of the two points as occurring
through diffusion, estimate again the time $t$ and compare your answer to that of (a).
**Problem 3:** Consider the gravitational stability of an infinite, isothermal cylinder of gas. The equation of hydrostatic balance is

\[-a_T^2 \frac{d \rho}{dR} - \rho \frac{d \Phi}{dR} = 0,\]

where \( \Phi \) is the gravitational potential. We conclude, as usual, that

\[\rho(R) = \rho_c \exp\left(-\Phi/a_T^2\right),\]

where \( \rho_c \) is the density at the central axis. Poisson’s equation now reads

\[\frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) = 4 \pi G \rho.\]

(a) Defining the nondimensional variables

\[\xi \equiv \frac{R \sqrt{4 \pi \rho_c G}}{a_T},\]

\[\psi \equiv \frac{\Phi}{a_T^2},\]

find the equation for \( \psi(\xi) \), as well as the appropriate boundary conditions.

(b) Find analytically \( \psi(\xi) \), and thereby the density profile \( \rho(R) \).

(c) Let \( M \) be the mass per unit length along the cylinder. The corresponding nondimensional quantity is

\[m \equiv \frac{GM}{a_T^2}.\]

Find analytically \( m(\xi) \).

(d) To analyze stability, plot \( m \) as a function of \( \rho/\rho_c \), the density contrast from center to edge. At what mass and density contrast does the stability transition occur?

**Problem 4:** C & C, Problem 36