(a) Hydrostatic equilibrium: \(-1/\rho \nabla P - \nabla \Phi_g = 0\). Poisson's equation: \(\nabla^2 \Phi_g = 4\pi G \rho\).

Also, we know that for isothermal material \(P = \rho \sigma T^2\), meaning we can write:

\[
\nabla^2 \Phi_g = -\nabla \left( \frac{a_T^2}{\rho^2} \nabla \rho \right) = 4\pi G \rho.
\]  

(15)

Then, because we're dealing with an infinite plane of gas we can rewrite \(\nabla\) as \(\partial/\partial z\). We get a second order differential equation for \(\rho\):

\[
\frac{a_T^2}{\rho^2} \left( \frac{\partial \rho}{\partial z} \right)^2 - \frac{a_T^2}{\rho} \frac{\partial^2 \rho}{\partial z^2} = 4\pi G \rho.
\]  

(16)

(b) Using the dimensionless variables given, \(\delta \equiv \rho/\rho_0\) and \(\zeta \equiv \sqrt{4\pi G \rho_0/a_T^2} z\), and that

\[
\frac{\partial}{\partial z} = \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta},
\]  

(17)

\[
= \sqrt{4\pi G \rho_0/a_T^2} \frac{\partial}{\partial \zeta},
\]  

(18)

we can rewrite the above differential equation as

\[
1 \left( \frac{\partial \delta}{\partial \zeta} \right)^2 - \frac{1}{\sqrt{4\pi G \rho_0/a_T^2}} \frac{\partial^2 \delta}{\partial \zeta^2} - \delta = 0.
\]  

(19)

The density of this slab at \(z = 0\) is \(\rho_0\) therefore \(\delta(\zeta = 0) = 1\). In order to get the other boundary equation we can integrate \(d^2 \Phi_g/dz^2\) from \(z = -\epsilon\) to \(z = \epsilon\) from which we get

\[
\frac{d\phi}{dz} \bigg|_{-\epsilon}^{+\epsilon} = 4\pi G \int_{-\epsilon}^{+\epsilon} \rho dz.
\]  

(20)

Since \(\rho(0) = \text{finite}\), then \(d\phi_g/dz(0) = 0\). From this we see that \(\rho'(0) = 0\) and so \(\delta'(0) = 0\), which gives our second boundary equation to solve this differential equation.
(c) We can solve the above equation. First, we rewrite the equation in terms of $\ln \delta$:

$$\frac{d^2 \ln \delta}{d\zeta^2} = -\delta. \quad (21)$$

Then we multiply both sides by $d \ln \delta/d\zeta$ to get

$$\frac{d \ln \delta}{d\zeta} \frac{d^2 \ln \delta}{d\zeta^2} = -\delta \frac{d \ln \delta}{d\zeta} = -\frac{d\delta}{d\zeta}, \quad (22)$$

which can be rewritten

$$\frac{1}{2} \frac{d}{d\zeta} \left( \frac{d \ln \delta}{d\zeta} \right)^2 = -\frac{d\delta}{d\zeta}. \quad (23)$$

We can integrate the above equation to get

$$\frac{1}{\delta^2} \left( \frac{d \delta}{d\zeta} \right)^2 = 2(1 - \delta), \quad (24)$$

where we have chosen our constant of integration in order to meet the boundary conditions. If we then rearrange and integrate the above equation we find

$$\int_1^\delta \frac{d\delta}{\delta \sqrt{1 - \delta}} = \sqrt{2}\zeta. \quad (25)$$

If we try $\delta = \text{sech}^2 \theta$, find that the above equation becomes simply

$$\theta = \zeta/\sqrt{2}. \quad (26)$$

Thus, the solution to our differential equation can be written as

$$\delta = \text{sech}^2 (\zeta/\sqrt{2}). \quad (27)$$

Transforming this equation back in terms of $\rho$ we find

$$\rho(z) = \rho_0 \text{sech}^2 \left( \sqrt{\frac{2\pi G \rho_0}{a_T^2}} z \right). \quad (28)$$
Problem 2

(a) We know that \( U = 3/2 \int P d^3 x \) and that for a SIS \( P = \rho a_T^2 \), therefore

\[
U = \frac{3}{2} \int \rho a_T^2 d^3 x.
\]  

(29)

However, \( a_T \) does not vary spatially so we can take that outside the integral and \( \int \rho d^3 x = M_0 \) so that we now have the following expression for \( U \)

\[
U = \frac{3}{2} M_0 a_T^2.
\]  

(30)

Integrate over the density to find the mass (Eqn. 9.9 from the book), and using the fact that \( \rho = a_T^2/2\pi Gr^2 \) we find that \( a_T^2 = GM_0/2R_0 \), which we can then substitute into our expression for \( U \) above

\[
U = \frac{3}{4} \frac{GM_0^2}{R_0}.
\]  

(31)

(b) To calculate \( f \) we must use the virial theorem, which in this case reads as \( 2U + \mathcal{W} = \mathcal{P} \). We already know \( U \) so we are left to solve for \( \mathcal{P} \), which is given as \( \mathcal{P} \equiv \int P \mathbf{r} \cdot \mathbf{v} d^3 x \). Thus,

\[
\mathcal{P} = 4\pi P_0^3 R_0.
\]  

(32)

The pressure is being evaluated on the surface of the sphere, and using our previous equation to relate the pressure to the density and evaluating at \( R_0 \) we find

\[
P(R_0) = a_T^4/2\pi GR_0^2
\]  

and substituting into the equation above we get

\[
\mathcal{P} = \frac{4\pi R_0^3 a_T^4}{2\pi GR_0^2} = \frac{GM_0^2}{R_0}.
\]  

(33)

Using this result, and our previous expression for \( U \) we find that

\[
\mathcal{W} = -\frac{GM_0^2}{R_0},
\]  

(34)

and hence \( f = 1 \).

c) Using equation (9.5a) and equation (9.8) we find

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_g}{dr} \right) = 4\pi G \rho,
\]  

(35)

and after substituting \( \rho(r) = a_T^2/4\pi Gr^2 \) and canceling the appropriate terms and differentiating we get

\[
\frac{d}{dr} \left( r^2 \frac{d\Phi_g}{dr} \right) = 2a_T^2.
\]  

(36)

Multiplying each side of this equation by \( dr \) and integrating we find

\[
r^2 \frac{d\Phi_g}{dr} = 2a_T^2 r,
\]  

(37)

and if we separate variables and integrate we find

\[
\Phi_g(r) = 2a_T^2 \ln r + C,
\]  

(38)
where \( C \) is a constant of integration. We want the potential to be smooth over the boundary, and outside the sphere it should look like \(-GM_0/r\). Therefore, we select \( C \) such that \( \Phi_g R_0 = -GM_0/R_0 \) which yields

\[
\Phi_g(r) = 2a_T^2 \ln(r/R_0) - GM_0/R_0.
\]  

(39)

(d) Using the given equation for \( W \),

\[
W = \frac{1}{2} \int \rho \Phi_g \, d^3 x,
\]

\[
= \frac{1}{2} \int_0^{R_0} \frac{a_T^2}{2\pi G r^2} [2a_T^2 \ln(r/R_0) - GM_0/R_0] 4\pi r^2 \, dr,
\]

\[
= \frac{2a_T^4}{G} \int_0^{R_0} \ln(r/R_0) \, dr - M_0 a_T^2.
\]

(40)  
(41)  
(42)

From here we see that the integral can be solved if we substitute \( x = r/R_0 \), in which case we are left with

\[
W = -\frac{2a_T^4 R_0}{G} - M_0 a_T^2,
\]

(43)

and using \( a_T^2 = GM_0/2R_0 \) we find

\[
W = -\frac{GM_0^2}{R_0}.
\]

(44)

Therefore, as we already saw in part (b) \( f = 1 \).
Problem 3

a)
From (3.16) $2U + W + M = 0$. Using, $U = \frac{3}{2}Ma_T^2$, $W = -\frac{GM^2}{R}$ and $M = \frac{R^2}{8\pi} \frac{4}{3} \pi R^3$, we have,

\[
\frac{B^2 R^3}{6} = \frac{GM^2}{R} - 3Ma_T^2
\]

\[
\Rightarrow B = \left( \frac{6GM^2}{R^4} - \frac{18Ma_T^2}{R^3} \right)^\frac{1}{3}
\]

b)
Taking $U = 0$,

\[
\frac{B^2 R^3}{6} = \frac{GM^2}{R}
\]

\[
\Rightarrow M_\Phi = \left( \frac{B^2 R^4}{6G} \right)^\frac{1}{2} = \frac{\Phi_d}{\pi} \frac{1}{\sqrt{6G}}
\]

using, $\Phi_d = \pi R^2 B$ for the magnetic flux. Evaluating the constants,

\[
M_\Phi = 0.13 \frac{\Phi_d}{\sqrt{G}}
\]

This is very close to equation (9.58), which gives a prefactor of 0.12.
c)

Going back to the virial theorem:

\[
\frac{B^2 R^3}{6} = \frac{GM^2}{R} - 3Ma_T^2
\]

Substituting in \( M = \frac{4}{3}\pi R^3 \rho \) in the third term and multiplying through by \( R/G \),

\[
\frac{B^2 R^4}{6G} = M^2 - \frac{4\pi}{G} R^4 \rho \alpha_T^2
\]

Using our equation for \( M_\phi \) from part b) and factoring out the third term,

\[
M_\phi^2 = M^2 - 4\pi R^4 \rho^{4/3} \left( \frac{a_T^3}{\rho^{1/2} G^{3/2}} \right)^{2/3}
\]

Substituting in, \( M_J = \frac{a_T^3}{\rho^{1/2} G^{3/2}} \),

\[
M_\phi^2 = M^2 - 4\pi \left( \rho R^2 \right)^{4/3} M_J^{2/3}
\]

which becomes,

\[
M_\phi^2 = M^2 - 4\pi \left( \frac{3}{4\pi} \right)^{4/3} M^{4/3} M_J^{2/3}
\]

The numeral factor is close to unity, so we have approximately

\[
M_\phi^2 = M^2 - M^{4/3} M_J^{2/3}
\]

Dividing through by \( M^2 \),

\[
\left( \frac{M_\phi}{M} \right)^2 = 1 - \left( \frac{M_J}{M} \right)^{2/3}
\]

We now identify \( M \) as the critical mass \( M_{crit} \). We find

\[
\frac{M_J}{M_{crit}} = \left[ 1 - \left( \frac{M_\phi}{M_{crit}} \right)^2 \right]^{3/2}
\]

\[
\Rightarrow M_{crit} = M_J \left[ 1 - \left( \frac{M_\phi}{M_{crit}} \right)^2 \right]^{-3/2}
\]
d)

If $M_{\text{crit}} \approx M_{\Phi} + M_{BE}$, then $\frac{M_{\text{crit}}}{M_{\Phi}} \approx 1 + \frac{M_{BE}}{M_{\Phi}}$. Dividing our answer from part c) by $M_{\Phi}$ and equating $M_J$ and $M_{BE}$,

$$\frac{M_{\text{crit}}}{M_{\Phi}} = \frac{M_{BE}}{M_{\Phi}} \left[ 1 - \left( \frac{M_{\text{crit}}}{M_{\Phi}} \right)^{-2} \right]^{-3/2}$$

If we try out a value for $\frac{M_{BE}}{M_{\Phi}}$, the corresponding value of $\frac{M_{\text{crit}}}{M_{\Phi}}$ (from the above equation) should be approximately equal to $1 + \frac{M_{BE}}{M_{\Phi}}$ in order for equation 9.57 to hold. At some point, this relation will break down.

<table>
<thead>
<tr>
<th>$M_{BE}/M_{\Phi}$</th>
<th>$M_{\text{crit}}/M_{\Phi}$</th>
<th>$1 + M_{BE}/M_{\Phi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>1.07</td>
<td>1.05</td>
</tr>
<tr>
<td>0.10</td>
<td>1.12</td>
<td>1.10</td>
</tr>
<tr>
<td>1.0</td>
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<td>2.0</td>
</tr>
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<td>5.0</td>
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<td>6.0</td>
</tr>
<tr>
<td>8.0</td>
<td>8.18</td>
<td>9.0</td>
</tr>
<tr>
<td>10.0</td>
<td>10.16</td>
<td>11</td>
</tr>
</tbody>
</table>

By the time we get to $M_{BE}/M_{\Phi} = 10$, the difference between $M_{\text{crit}}/M_{\Phi}$ and $1 + M_{BE}/M_{\Phi}$ is about 8%.
Problem 4

a) If we set the direction of \( B_0 \) as \( \hat{z} \), then
\[
k_z = k \cos(\theta_B)
\]
The dispersion relation becomes
\[
\omega = k_z V_A
\]
Thus
\[
[V_{\text{group}}]_x = \frac{\partial \omega}{\partial k_z} = V_A
\]
while \( [V_{\text{group}}]_y = [V_{\text{group}}]_z = 0 \)

Thus energy travels along the magnetic field at speed \( V_A \).

b) Begin with equation (9.66):
\[
-\omega \delta \vec{B} = (\vec{k} \cdot \vec{B}_0) \delta \vec{u} - (\vec{k} \cdot \delta \vec{u}) \vec{B}_0
\]

For a transverse wave, \( \vec{k} \cdot \delta \vec{u} = 0 \). Setting \( \vec{k} \cdot \vec{B}_0 = k_z B_0 \),
\[
-\omega \delta \vec{B} = k_z B_0 \delta \vec{u}
\]
\[
\Rightarrow \delta \vec{u} = \frac{\omega}{k_z B_0} \delta \vec{B}
\]

and \( \delta \vec{u} \) and \( \delta \vec{B} \) are antiparallel. Using \( \omega = k_z V_A \), we have
\[
\frac{\delta \vec{u}}{V_A} = -\frac{\delta \vec{B}}{B_0}
\]
which reproduces equation (9.89).
\[ \frac{E_{\text{mag}}}{E_{\text{kin}}} = \frac{|\delta B|^2}{8\pi} \frac{2}{|\delta u|^2 \rho_0} \]
\[ = \frac{\delta B^2}{8\pi} \frac{2B_0^2}{\delta B^2 V_A^2 \rho_0} \]
\[ = \frac{B_0^2}{4\pi \rho_0 V_A^2} \]
\[ = 1 \]

from the definition of \( V_A \).

d)
Plugging \( \delta j \) into \( f \),
\[ f = \frac{\delta j \times B_0}{c} \]
\[ = \frac{1}{c} \left[ \frac{c}{4\pi} (\nabla \times \delta B) \times B_0 \right] \]
\[ = \frac{-1}{4\pi} [\nabla (B_0 \cdot \delta B) - (B_0 \cdot \nabla) \delta B] \]

where we have expanded the triple cross product and used \( \nabla \cdot B_0 = 0 \). From equation (9.80), \( \delta u \cdot \hat{k} = 0 \) implies that \( \delta u \cdot B = 0 \). Thus, from part b) \( \delta B \cdot B_0 = 0 \), and
\[ f = \frac{(B_0 \cdot \nabla) \delta B}{4\pi} \]

which is the form of magnetic tension from equation (3.6).
Problem 5

a)

Starting with equation (10.3):

$$v_{drift} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi nm_n [m_n < \sigma_{in} u_i >]}$$

To calculate the triple cross-product, we note that $\mathbf{B}$ is in the $x$-direction, and there are no $x$- or $y$-gradients. Thus,

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{\nabla B^2}{2} (-\hat{z})$$

$$\Rightarrow v_{drift} = -\frac{\nabla B^2}{8\pi} \frac{1}{\rho \rho_t \gamma}$$

where we have used $nm_n = \rho$. Our slab is partially supported against self-gravity by magnetic pressure. Hence, $\frac{B^2}{8\pi}$ decreases away from the midplane.
Thus $v_{drift}$ is in the $+\hat{z}$ direction. Since $v_{drift} = u_i - u$, the neutrals are moving toward the midplane, in the $-\hat{z}$ direction.

b)

First taking the convective derivative of $B/\rho$ and plugging in continuity:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial z}$$

$$\frac{D(\rho B)}{Dt} = \frac{1}{\rho} \frac{\partial B}{\partial t} - \frac{B}{\rho^2} \left(-\rho \frac{\partial u}{\partial z} - u \frac{\partial \rho}{\partial z}\right) + \frac{u \partial B}{\rho} - \frac{u B \partial \rho}{\rho^2} \frac{\partial \rho}{\partial z}$$

$$= \frac{1}{\rho} \frac{\partial B}{\partial t} + \frac{B}{\rho} \frac{\partial u}{\partial z} + \frac{u B}{\rho} \frac{\partial \rho}{\partial z}$$

Equation (10.4) reads:

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \nabla \times (v_{drift} \times B)$$

Expanding the triple cross product, and using $\vec{\nabla} \cdot \vec{B} = 0$,

$$\frac{1}{\rho} \frac{\partial B}{\partial t} = \frac{u}{\rho} \frac{\partial B}{\partial z} - \frac{B}{\rho} \frac{\partial u}{\partial z} - \frac{v_{drift} \partial B}{\rho} - \frac{B \partial v_{drift}}{\rho}$$

Using the previous result

$$\frac{D(B/\rho)}{Dt} = -\frac{1}{\rho} \frac{\partial}{\partial z}(B v_{drift})$$

c)

From the definition of $\sigma$,

$$\frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{B^2}{8\pi}\right) = \frac{\partial}{\partial \sigma} \left(\frac{B^2}{8\pi}\right)$$

Plugging this back into $v_{drift}$ from a);

$$v_{drift} = -\frac{1}{\gamma \rho_i \frac{\partial}{\partial \sigma} \left(\frac{B^2}{8\pi}\right)}.$$
First we need to convert from derivatives in terms of $z$ to $\sigma$. Using continuity:

$$\left( \frac{\partial \sigma}{\partial t} \right) = \int \frac{\partial \rho}{\partial t} dz = - \int \frac{\partial (\rho u)}{\partial z} dz = -\rho u$$

Since $u(0) = 0$. Next, find $\left( \frac{\partial}{\partial t} \right)_z$ in terms of $\left( \frac{\partial}{\partial t} \right)_\sigma$. Writing out the derivative explicitly,

$$\left( \frac{\partial}{\partial t} \right)_\sigma dt + \left( \frac{\partial}{\partial \sigma} \right)_t d\sigma = \left( \frac{\partial}{\partial z} \right)_t dz + \left( \frac{\partial}{\partial t} \right)_z dt$$

Holding $z$ constant:

$$\left( \frac{\partial}{\partial t} \right)_\sigma = \left( \frac{\partial}{\partial t} \right)_z - \left( \frac{\partial}{\partial \sigma} \right)_t \left( \frac{\partial \sigma}{\partial t} \right)_\sigma$$

$$\Rightarrow \left( \frac{\partial}{\partial t} \right)_z = \left( \frac{\partial}{\partial t} \right)_\sigma - (\rho u) \left( \frac{\partial}{\partial \sigma} \right)_t$$

Finally, we need to find $\left( \frac{\partial}{\partial \sigma} \right)_t$ in terms of $\left( \frac{\partial}{\partial \sigma} \right)_t$

$$\left( \frac{\partial}{\partial \sigma} \right)_t = \left( \frac{\partial}{\partial \sigma} \right)_t \left( \frac{\partial \sigma}{\partial z} \right)_t = -\rho \left( \frac{\partial}{\partial \sigma} \right)_t$$

We can now write out the convective derivative in terms of $\sigma$,

$$\frac{D(B/\rho)}{Dt} = \left( \frac{\partial (B/\rho)}{\partial t} \right)_z + u \left( \frac{\partial (B/\rho)}{\partial z} \right)_t$$

$$= \left( \frac{\partial (B/\rho)}{\partial t} \right)_\sigma + \left( \frac{\partial (B/\rho)}{\partial \sigma} \right)_t (-\rho u) + \left( \frac{\partial (B/\rho)}{\partial \sigma} \right)_t \rho u$$

$$= \left( \frac{\partial (B/\rho)}{\partial \sigma} \right)_\sigma$$

The convective derivative is just the time derivative with respect to $\sigma$. Putting this together with our result from parts b) and c),

$$\left( \frac{\partial (B/\rho)}{\partial t} \right)_\sigma = -\frac{1}{\rho} \left[ \frac{\partial}{\partial z} (B v_{dr}) \right]_t = - \left[ \frac{\partial}{\partial \sigma} (B v_{dr}) \right]_t$$

$$\Rightarrow \left( \frac{\partial (B/\rho)}{\partial t} \right)_\sigma = \frac{1}{\gamma \rho} \left[ B^2 \frac{\partial B}{\partial \sigma} \right]$$
e)

We can estimate the characteristic diffusion time using our answer from part d):

$$\frac{B/\rho}{t} \sim \frac{B^3}{\gamma \rho \sigma^2}$$

$$\Rightarrow t_{diff} \sim \frac{\gamma \sigma^2 \rho}{B^2 \rho}$$

From equation (8.36), $\rho \sim C \rho^{1/2}$. Using this,

$$\Rightarrow t_{diff} \sim \frac{\gamma \sigma^2 C}{B^2 \rho^{1/2}}$$

We can further reduce our expression for $t_{diff}$ by first noting that, for hydrostatic balance,

$$g \sim \frac{GM}{R^2} \sim G \sigma$$

Thus, the thermal pressure is

$$P \sim \rho gh \sim \rho G \sigma h \sim G \sigma^2$$

Since $P = \rho a_T^2$, then approximate equality of thermal and magnetic pressure gives

$$B^2 \sim G \sigma^2 \text{ and } \rho a_T^2 \sim G \sigma^2$$

Substituting these relations into our expression for $t_{diff}$,

$$t_{diff} \sim \frac{\gamma \sigma^2 C a_T}{G \sigma^2 G^{1/2} \sigma}$$

$$\sim \frac{\gamma C a_T}{G^{1/2} G \sigma}$$

Thus,

$$t_{diff} \approx \frac{a_T}{G \sigma}$$

where $n = \frac{\gamma c}{G^{1/2}}$ is a dimensionless number.
Problem 6

(a) If we imagine that a parcel of gas is falling in from $R(t)$ at the freefall velocity $v_H(r)$ for its entire journey, then

$$
\Delta t = \int_{R(t)}^{r_0} \frac{dr}{v_H(r)}
= \frac{-1}{\sqrt{GM_\ast}} \int_{R(t)}^{r_0} r^{1/2}dr
= \frac{2R^{3/2}}{3\sqrt{GM_\ast}}.
$$

(b) We'll make some simplifying assumptions about the protostar and rarefaction wave. If we assume a constant accretion rate based on the cloud freefall speed,

$$
M_\ast = \dot{M}_t = \frac{a_1^3}{G} t.
$$

We then parametrize $R$ in terms of the sound speed and a factor $f$:

$$
R = f a_\Omega t.
$$

Plugging these definitions into (6), we find

$$
\frac{\Delta t}{t} = \frac{2}{3} f^{3/2}.
$$

This is negligible only for small values of $f$; that is, the steady-state assumption is only valid well inside the rarefaction wave.

(c) Equation (10.34) tells us that

$$
\rho(r) = \frac{1}{4\pi \sqrt{2}} G^{-1/2} M_\ast^{-1/2} \dot{M} r^{-3/2}.
$$

Calculating $\Delta M$ is an easy integral:

$$
\Delta M = \int_0^R 4\pi r^2 \rho(r)dr
= (2GM_\ast)^{-1/2} \dot{M} \int_0^R r^{1/2}dr
= \frac{2}{3} (2GM_\ast)^{-1/2} \dot{M} R^{3/2}.
$$

(d) As before, let $M_\ast = a_0^3 t / G$ and $R = f a_\Omega t$. Then, substituting,

$$
\frac{\Delta M}{M_\ast} = \frac{\sqrt{2}}{3} f^{3/2}.
$$

As before, this is only negligibly small if $f \ll 1$, well inside the rarefaction wave.