1 Public vs. Private

The University of California prides itself on being a public school system, but in recent years the fraction of its budget supported by the State of California has dwindled to $\sim 10\%$. Twenty-five years ago, that fraction was $\sim 50\%$.

How much more in annual taxes would the average Californian have to pay to restore the state funding fraction back to $\sim 50\%$?

People are money. At Berkeley, there are about $2e3$ faculty each of which are paid $1e5$ K per year; $1e3$ staff each of which are paid $1e5$ K; and $1e4$ graduate students each of which are paid $3e4$ K. That’s a total salary cost of $6e8$ K. There are 10 other UC schools but maybe only 4 with the size equivalent of Berkeley, so we’ll say the total annual operating budget for UC is $6e8 \times 4 \sim 2e9$. Take $50\%$ as the problem states to get $1e9$.

There are $3e8$ people in the US and about 10% of them live in the Golden State, or $3e7$ Californians. But only about 50% of these are taxpayers, so $1.5e7$ state taxpayers, each of which would pay $70$ per year to fund half of the UC system.

This estimate accords with the Op-Ed piece written by James Vernon on Nov 15 2014 stating that the median individual taxpayer would have to shell out $50$ per year to restore funding levels back to 2000–2001.

This seems a modest sum and it is sad that California can’t muster enough will to invest in higher education.
A violin has two f-shaped holes ("f-holes") carved into its top, of length \( \sim 10 \) cm and width \( \sim 7 \) mm. The thickness of the wood is about 2 mm.

Estimate the fundamental frequency (in Hz) at which air in this box vibrates.

From class, the fundamental frequency of a Helmholtz resonator is

\[
\nu_0 = \frac{c_s}{2\pi} \sqrt{\frac{A}{V_0 l}}
\]

where \( c_s = 350 \) m/s, \( A = 10 \times 0.7 \) cm\(^2\), \( t = 2.7 \) mm, \( d = 7 \) mm, and \( V_0 = 40 \times 15 \times 3 \) cm\(^3\) or 1.6 liters.

The tricky part is estimating the effective thickness \( l \) of the "air plug" that is vibrating near the f-hole; 2 mm is an underestimate because the hole dimensions are much larger than that; and it’s probably not as big as 10 cm (the largest lengthscale of the hole); we’ll take a geometric average of 10 cm and 0.7 cm, to get 2 cm. That seems plausible. Anyway it comes in as the square root so we shouldn’t get too stressed about it.

Plugging in, we get \( \nu_0 \sim 400 \) Hz, which is close to the measured answer of 300 Hz.

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1This was the reply of pianist Arthur Rubenstein when he was asked by a New York City tourist, “How does one get to Carnegie Hall?”
3 It’s Always Greener on the Other Side of the Fence

A demonstration on Cal Day: A 5-gallon transparent drum is upturned over a patch of grass in bright sunshine. Estimate the time it takes for the grass to significantly reduce the CO$_2$ within the drum.

CO$_2$ constitutes about 350 ppmv (parts per million by volume) of the atmosphere.

5 gallons is 0.02 m$^3$, so I’ll guess that the drum is 30 cm tall with an opening of radius $R = 15$ cm. The mass of CO$_2$ in the drum is

$$m_{CO_2} = \rho_{CO_2} V_{CO_2} \approx 350 \cdot 10^{-6} \rho_{CO_2} V.$$  

The density of CO$_2$ is 2 kg/m$^3$, so

$$m_{CO_2} \approx 10^{-8} \text{ g}.$$  

We need to figure out how long it takes the grass under the drum to consume that much carbon dioxide. How fast does grass grow? From my lawn care days, I remember that grass grows about $h = 2$ cm or so in a week. The density of grass is about 0.5 g cm$^{-3}$ (it floats), and the blades can be modeled as cylinders with $r = 1$ mm spaced with maybe $\sigma = 10$ blades per square centimeter. In one minute, therefore, the mass of grass that grows under the drum is

$$\dot{m} = \frac{\rho_g (\pi r^2) h \sigma (\pi R^2)}{7 \cdot 24 \cdot 60} \approx 2 \cdot 10^{-8} \text{ g s}^{-1}.$$  

Now, plants are mostly (90%) water, and maybe half of the remainder is carbon, so the rate at which carbon is removed from the tank is that times $f = 0.1 \cdot 0.5 \approx 10^{-9} \text{ g s}^{-1}$ (don’t worry about stoichiometry - this is OOM). The time it takes for the CO$_2$ to get depleted, then, is

$$\frac{m}{\dot{m}} \approx 10 \text{ minutes}.$$  

Plausible, considering this was a demonstration performed at Cal Day.
4 Riddle of the Sphinx

The Great Pyramid of Giza, built to entomb Egyptian pharaoh Khufu, stands about 150 meters tall. Its base is a square of length 230 m.

Estimate the number of workers and the number of years required to build the Great Pyramid. Express your answer in worker-years.

The volume of the pyramid is \( \frac{1}{3} Ah \) and its center of mass is \( \sim \frac{1}{4} \) of the way up. The gravitational potential energy required to assemble the pyramid, then, is

\[
P.E. = \rho V g \frac{h}{4} = \frac{1}{12} \rho g Ah^2
\]

We should also include the energy required to transport all those stones from the quarry to the construction site. The coefficient of rolling friction might be \( \mu \sim 0.1 \) (see previous problem set on car mileage) and the distance \( D \) to the quarry might be 10 km. The work done against friction is therefore

\[
W = \rho V g \mu D = \frac{1}{3} \rho g Ah \mu D
\]

The total energy to build the pyramid, then, is

\[
E = \frac{1}{12} \rho g Ah (h + 4 \mu D)
\]

Notice that a lot more work is done moving the rocks by 10 km than by lifting them up by \( h \sim 150 \) m, by a factor of \( 4 \mu D/h \sim 4000/150 \sim 30 \).

What about the workers? A human’s base metabolic rate is 100 W, and if we work hard we’re capable of exerting maybe 5 times that for an extended period. The people who built the pyramids, however, were probably weak and malnourished, so let’s say they were only capable of operating at twice the normal BMR. That’s an extra 100 W with which to do work, and factoring in a 25% muscle efficiency and a 50% efficiency for sleeping tells us that the power available to build the pyramid is

\[
P = N \times 10 W.
\]

The pyramid was built during the reign of one guy, so let’s say that it took maybe \( T = 20 \) years to build. The number of workers \( N \) required would then be

\[
N \sim \frac{\frac{1}{12} \rho g Ah (h + 4 \mu D)}{10 WT} \approx 10,000 \text{ workers}
\]

This is in line with estimates made by modern day Egyptologists.

Our final answer is \( 1e4 \times 20 \sim 2e5 \text{ worker-years} \).
5 Saving Private Ryan

To what minimum depth of water would a World War II frogman (combat swimmer) need to dive to avoid being killed by machine gun bullets fired from the beaches of Normandy?

Pretty much pressure drag, \( F_D \sim (1/2) \rho v^2 r^2 \). The time for your velocity to reduce by order \( e \) is \( \Delta t \sim mv/F_D \sim m/(1/2)\rho ho v r^2 \). A nice way to remember this is to recognize that \( \Delta t \) is the time it takes for you to run into your own mass (modulo factors of 2 and non-sphericity of the object). Because the bullet is denser than water by a factor of \( \sim 8 \), that means the bullet needs to run into about 8 times its own length before it slows down by a factor of \( e \). A bullet is maybe 1 cm long, so that’s 8 cm.

But that’s just to reduce the bullet velocity by a factor of 2–3. To not hurt, the bullet must slow down a lot more than that. Figure it’s initially travelling at close to the speed of sound, or 340 m/s, and that a “safe” velocity is less than 10 cm/s. So that requires a factor of 3000 reduction or about 9 e-folds. So our final answer is \( 8 \text{ cm} \times 9 \sim 72 \text{ cm} \). Basically \([ \text{a meter}] \) should suffice.

This problem has been addressed in not one but two Mythbusters episodes. They got the same result for 9 mm rounds, although interestingly with more powerful guns, the bullet often shatters upon hitting the water, so more powerful guns actually don’t require as much water for safety. It goes to show you what we said in class, that “liquids are pretty much solids”.
6 Raindrops Keep Falling on My Head

Derive an order-of-magnitude analytic formula for what sets the maximum sizes of raindrops. The more your formula depends on fundamental constants, the more credit will be awarded.

What’s holding the raindrop together is surface tension, which exerts a binding force of \( \sim \gamma \times 2\pi r \) where \( r \) is the raindrop size. What’s trying to split the raindrop apart is the aerodynamic drag force — really the fluctuating eddies in the marginally turbulent boundary layer around the raindrop — or \( \sim 1/2 \times \rho_{\text{air}} v^2 \pi r^2 \) (times \( C_D \) which is pretty much 1). So we can solve for \( r \sim 2\pi \gamma / (\rho_{\text{air}} v^2) \).

But we can go even more fundamental, by solving for \( v \). Figure the raindrop is at terminal velocity, so \( mg \sim \rho_{\text{air}} v^2 r^2 \). So \( \rho_{\text{air}} v^2 \sim mg/r^2 \sim (4/3) \pi \rho_{\text{water}} r g \sim 4\rho_{\text{water}} r g \). Plug this into the formula in the above paragraph — \( r \sim 2\pi \gamma / (4\rho_{\text{water}} r g) \) — and solve for \( r \sim (\gamma / \rho_{\text{water}} g)^{1/2} \). Which gives 2 mm which is about right.
7 Back to the Future

On Oct 21 2014, the Los Gatos, CA start-up Hendo announced that it has built the world’s first magnetically levitating skateboard: the Hendo Hoverboard.

On Oct 21 2015, Marty McFly, Jr. will step out of his time-traveling DeLorean and onto a levitating skateboard.

The Hendo Hoverboard only hovers above a conductive, non-ferrous floor—e.g., copper or aluminum. The hoverboard operates on the principle of Lenz’s Law. The bottom of the hoverboard has 4 “engines”—see attached Fig. 1 on the last page of this exam.

Each engine is composed of a spinning disc composed of alternating permanent magnets—see Fig. 2. The spinning magnets generate time-varying magnetic fields. These variable fields induce electrical currents (“eddy currents”) in the conductive floor. These eddy currents produce their own magnetic fields, which repel the magnetic fields of the hoverboard to produce lift.

FYI (you don’t have to literally use these), here are Maxwell’s equations in cgs units:

\[
\begin{align*}
\nabla \cdot E &= 4\pi \rho_e \\
\nabla \cdot B &= 0 \\
\nabla \times B &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial E}{\partial t} \\
\nabla \times E &= -\frac{1}{c} \frac{\partial B}{\partial t}
\end{align*}
\]

And in SI units:

\[
\begin{align*}
\nabla \cdot E &= \frac{1}{\epsilon_0} \rho_e \\
\nabla \cdot B &= 0 \\
\nabla \times B &= \frac{1}{\epsilon_0 c^2} \mathbf{J} + \frac{1}{c^2} \frac{\partial E}{\partial t} \\
\nabla \times E &= -\frac{\partial B}{\partial t}
\end{align*}
\]

where \(\epsilon_0 = 8.9 \times 10^{-12}\) (SI).

1 electron charge = \(4.8 \times 10^{-10}\) esu (cgs) = \(1.6 \times 10^{-19}\) Coulombs (SI)

\(10^4 \, G \) (cgs) = \(1 \, T\) (SI)

\(^2\)No knowledge of this movie is required to solve this problem.
(a) Estimate the strength of the magnetic field $B$ required to support both the board (which weighs 100 lbs) and its rider.

Magnetic pressure is $B^2/\pi$ so magnetic force is $(B^2/8\pi)\times\text{Area}$ where $\text{Area} \sim 4 \times \pi(3\text{ cm})^2$, taking each disc to have a radius of 3 cm. Set the magnetic force equal to the weight of gravity $mg$ using $m \sim 100$ kg (board + rider).

Solve for $B \sim 5000$ G $\sim 0.5$ T. This sounds right: the Hendo magnets are permanent neodymium magnets, and such good quality magnets have surface fields between 0.1–1 T. The magnets hover above the ground so closely that the field strength is pretty much the surface field strength (recall also the YouTube video we saw in class which featured an Englishman being levitated by a neodymium magnet).

(b) Estimate the rotation frequency $f$ (in rotations per minute = rpm) of each of the magnetized disks. The floor is $d \sim 1$ cm thick and composed of copper — which is not a perfect conductor.\(^3\)

According to Purcell’s cheat sheet, copper has a resistivity of $2 \times 10^{-6}$ ohm-cm (mixed SI-cgs units) $= 2 \times 10^{-8}$ ohm-m (SI) $= 2 \times 10^{-8}$ m/Siemens (SI) $= 2 \times 10^{-18}$ s (cgs).

We need the hoverboard magnets spinning fast enough that they induce strong enough eddy currents in the ground that those eddy currents can generate strong enough ground fields to repel the hoverboard fields. In other words, we need the ground field to be as strong as the hoverboard field of 5000 G (our answer in (a)) (and of course to have opposite sign).

How much eddy current do we need in the ground to generate a given ground field? Ampere’s Law tells us $4\pi J/c \sim \nabla \times B$ (forgetting the displacement current term in the cavalier spirit of OOM). Approximate $\nabla \times B \sim B/d$, where $d$ is the ground thickness. Then

$$J \sim \frac{Bc}{4\pi d}$$

Note that it may not be so obvious that the relevant lengthscale is the ground thickness $d$. But it is, because a faint memory of E&M reminds us that a current sheet produces a magnetic field above it; the strength of the field must depend on how much current runs through the sheet, which in turn must scale as $Jd$; remember that $J$ is current density, having units of charge per time per area; so we must have $B \propto Jd$ which is what the above equation says.

Now we use Faraday’s law of induction: $\nabla \times E \sim (1/c)\partial B/\partial t \sim (1/c)B/\Delta t$. The $\Delta t$ is what we want — the timescale for the field to change at a fixed position is $\Delta t \propto 1/f$. We can solve for $\Delta t$ if we know $E$. But we know $E$ — we know it because of Ohm’s Law, which relates $J$ to $E$. In grade school (OK, high school), we learn one form of Ohm’s Law: $I = V/R$. In grad school (and Astro 250), we learn the adult form of Ohm’s Law:

$$J = \sigma E = E/\rho$$

where $\sigma$ is the electrical conductivity, which is the inverse of the electrical resistivity $\rho$. So we can solve for

\(^3\)If it were a perfect conductor, you could observe the Meissner effect, which is the standard levitation trick in superconductivity.
\[ E \sim \rho J \sim \frac{\rho Bc}{4\pi d} \]

Plug this \( E \) into Faraday: \( \nabla \times E \sim \frac{E}{\Delta L} \sim \frac{\rho Bc}{(4\pi d)\Delta L} \sim \frac{(1/c)B}{\Delta t} \), where \( \Delta L \) is the lengthscale over which the electric field varies, which is pretty much the arclength of one of the “magnetic sectors” or pie slices into which the disc is divided (see Figure 2). We can solve this equation for the disc rotation velocity

\[ v \sim \frac{\Delta L}{\Delta t} \sim \frac{c^2 \rho}{4\pi d} \]

What we want is the rotation frequency \( \omega = v/r \) where \( r \sim 3 \text{ cm} \) is the disc radius:

\[ \omega \sim \frac{c^2 \rho}{4\pi d \times r} \]

Plugging in, we get \( \omega \sim 50 \text{ rad/s} \). There are \( 2\pi \) rad in one revolution and 60 seconds in 1 minute, so the desired rotation frequency \( f \sim 500 \text{ rpm} \).

(c) **Estimate the power requirement** \( P \) **(in Watts) of the hoverboard.**

The power requirement is pretty much that required to drive currents through the resistive floor. For a resistance \( R \) and current \( I \), the Ohmic power is

\[ P \sim I^2R \sim (J \times r \times d)^2 \times \frac{\rho \Delta L}{r \times d} \times N_{\Delta L} \times 4 \]

where \( N_{\Delta L} \) is the number of magnetic sectors or pie slices, and the 4 is for the 4 discs. We’re thinking of each pie slice as its own circuit, each of resistance \( \frac{\rho \Delta L}{(rd)} \) and current \( Jrd \).

Now \( N_{\Delta L} \Delta L \sim 2\pi r \), the full circumference of the disc. Also we can plug in for \( J \) using (b). So we get:

\[ P \sim \left( \frac{cB}{4\pi d} \right)^2 \rho rd \times 2\pi r \times 4 \sim \frac{(cB)^2}{2\pi} \rho r^2 d \]

Plugging in, we get \( P \sim 6000 \text{ W} \). It’s hard to imagine the engines are 100% efficient, but maybe it’s not so bad since the only other sources of loss are the resistance in the electric motor used to spin the discs, and friction in the discs themselves (aerodynamics and bearings). I’ll just round up to \( P \sim 10^4 \text{ W} \).

(d) **Estimate the battery lifetime** \( \Delta t \) **of the hoverboard (which weighs 100 lbs total).**

A previous problem set on energy storage in lithium-ion batteries gives 0.2 kW-hr / kg \( \sim 7 \times 10^9 \) erg/g for the battery energy density. We can also estimate this by saying each lithium lattice site stores \( \sim 1 \text{ eV} \) of energy using a bunch of complicated heavy metal electrolytes which might weigh \( \sim 100 \text{ m}_H \) per lattice site. That gives \( 10^{10} \text{ erg/g} \), which is pretty close to the problem set answer.

Say \( 1/3 \) of the weight of the hoverboard is batteries, so that’s a battery mass of \( (1/3) \times 50 \text{ kg} \sim 16000 \text{ g} \). Using the energy density above, that’s \( 10^{14} \text{ erg} \). Divide this energy by \( 10^4 \text{ W} \) to get a battery lifetime of \( 10^3 \text{ s} \) or \( 10 \text{ minutes} \). The Hendo hoverboard actually touts a ride time of 7 minutes.
Figure 1: The Hendo Hoverboard showing the 4 spinning disks on the bottom of the board.

Figure 2: Schematic of 1 spinning disk composed of alternating permanent magnets.