7B Math Review: Chain Rule

Casey Lam

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Chain Rule

Definition

If you have a composition of functions $f(g(x))$, then the derivative with respect to x is

$$(f(g(x)))' = f'(g(x))g'(x) \Leftrightarrow \frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

The basic intuition of the chain rule is that if f depends on g, and g depends on x, then to see how f changes when x is changed, you need to "go down the chain": see how f changes when g is changed, then see how g changes when x is changed, and then multiply them together.

If your inner mathematician will forgive me, I prefer Leibniz notation (the right hand side) over Lagrange notation (the left hand side), because you can see how the dg's "cancel". (Of course, the dg's don't *really* cancel because $\frac{df}{dg}$ and $\frac{dg}{dx}$ are not fractions! But this is a nice mnemonic we can use to remember the chain rule.)

Practice questions

1. Q: If x = x(t), what is the derivative of x^5 with respect to t?

   A: $\frac{d}{dt}(x^5) = 5x^4\dot{x}$ (Notation: an overdot is used to symbolize a derivative with respect to time.)

2. Q: If y = y(t), what is the second derivative of $y^3 + y^2 + y$ with respect to t?

   A: $\frac{d}{dt}(y^3 + y^2 + y) = 3y^2\dot{y} + 2y\dot{y} + \dot{y} \implies \frac{d^2}{dt^2}(y^3 + y^2 + y) = 6y\dot{y}^2 + 3y^2\ddot{y} + 2\dot{y}^2 + 2y\ddot{y}^2 + \ddot{y}$

   The key to this one is to remember the product rule! Also a good thing to brush up on if you've forgotten.

Other things to note: make sure your units work out! For example, in the first example, you have 5 x's in the numerator and one t in the denominator. On the right side, you also have 5 x's in the numerator and one t in the denominator (remember that $\dot{x} = \frac{dx}{dt}$.) Dimensional analysis for math!