Astro/Phys 202: Astrophysical Fluid Dynamics September 17 2025

C.–P. Ma

Problem Set 2

Due 6pm Friday September 26

0. **Reading**: Finish Chapter 13.1-13.6 of Thorne and Blandford

1. **Polytropic Stellar Models I: Analytic Solutions**

(a) Find the analytical solution $\theta(\xi)$ to the Lane-Emden equation for a non-rotating spherical star with a polytropic index $n = 0$. Find the value of ξ at the star's surface.

(b) Repeat (a) for $n = 1$.

(c) Plot on a **single** figure the radial profile θ vs ξ for a polytropic star of $n = 0, 1$, and 5. Comment on the general trend as n increases. Attach your plotting script to show work.

2. **Polytropic Stellar Models II: Some Physical Numbers**

Consider a non-rotating spherical star of polytropic index n (with $n < 5$), mass M, and radius R.

(a) Show that the ratio of the star's central density to the mean density is

$$\frac{\rho_c}{\bar{\rho}} = \frac{\xi_{\rm max}^3}{3I_n}\,, \qquad I_n \equiv \int_0^{\xi_{\rm max}} \theta^n \xi^2 d\xi\,. \tag{1}$$

The Sun follows approximately the $n = 3$ polytropic model. For $n = 3$, the Lane-Emden equation does not have an analytic solution, but it can be solved numerically to show that $\xi_{\rm max} = 6.9$, and $I_3 = 2.02$.

(b) For $n = 3$, show that the star's mass is independent of its central density and depends on only constants and the coefficient K. Compute the value of K for the Sun.

(c) Using information from (a) and (b), compute the value of the Sun's (i) mean density in g/cm^3, (ii) central density in g/cm^3, (iii) central pressure, and (iv) central temperature T_c (assuming an ideal gas with mean molecular weight of 0.6). How does your T_c compare with the value of 1.5×10^7 Kelvin obtained from numerical solution of the full set of stellar structure equations for a $1M_\odot$ star?

3. **Giant Gas Planets**

Jupiter and Saturn are both comprised of H-He fluid that is well approximated by a polytropic index of $n = 1$, that is, $P = K\rho^2$ with the same constant K. Use the fact that $M_J = 1.9 \times 10^{27}$ kg, $M_S = 5.7 \times 10^{26}$ kg and $R_J = 7 \times 10^4$ km to estimate the radius of Saturn. Using the analytic solution to the Lane-Emden equation with $n = 1$ in Problem 1(b) above, compute the central densities of Jupiter and Saturn.