Astro/Phys 202: Astrophysical Fluid Dynamics September 26 2025

C.–P. Ma

Problem Set 3

Due 6pm Friday October 3

0. **Reading**: Sections 16.1, 16.2, 16.5 of Thorne and Blandford

1. **Capillary Waves**

When the wavelength of gravity waves is very short, we must include the effects of surface tension on the surface boundary conditions. This tension produces on the fluid's surface a net downward force per unit area $-Td^2\xi_z/dx^2$. The associated waves are called capillary waves.

(a) Show that the surface boundary condition in this case has the same form as the one derived in class, if the acceleration of gravity g is replaced by an effective acceleration $g_{\rm eff}$ as:

$$P_1 = \rho_0 \xi_z g_{\rm eff} , \qquad g_{\rm eff} = g + \frac{Tk^2}{\rho_0} . \tag{1}$$

(b) Write down the dispersion relation, and the waves' phase velocity in the deep water limit. Show that the surface tension is important for waves with wavelengths $\lambda \ll \lambda_c$, and find an expression for the critical wavelength λ_c. Compute λ_c (in cm). (Surface tension of water molecules is approximately $71\,{\rm erg/cm^2}$.) We derived in class that for surface gravity waves (in deep water), longer waves travel faster. What about capillary waves?

2. **Shock Jump Conditions**

(a) From the jump conditions we derived in class, show that $x \equiv v_2/v_1$ is either unity (i.e. no shocks) or

$$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)}{(\gamma-1) + 2/M_1^2} \tag{2}$$

for the ratio of the post-shock (downstream) and pre-shock (upstream) densities and velocities, where M_1 is the pre-shock Mach number.

(b) Show that the pressure and velocities can also be re-expressed entirely in terms of M_1 as

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} , \qquad M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)} . \tag{3}$$

(c) From eq. (3), show that if $M_1 > 1$, then $M_2 < 1$. That is, the post-shock is always subsonic. (Recall γ ranges from 1 to infinity.)