Astro/Phys 202: Astrophysical Fluid Dynamics November 8 2024

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Problem Set 8

Due 6pm Friday November 22 2024

Reading: Sec. 13.7 of Thorne & Blandford; Ch. 7 of Balbus Hydrodynamics

0. **Existence and Smoothness (or Not!) of the Navier-Stokes Equation**

The existence and smoothness of solutions to the Navier-Stokes equation in 3D is one of seven Millennium problems posted here. Read the official description of this problem. This lecture by Atiyah is quite elegant. Write down any thoughts (or solution - extra credit!) you may have.

1. **Stokes Flow**

In this problem, you are led through the derivation for the drag force on a sphere moving through a viscous fluid. This is a classic fluid dynamics problem first solved by Stokes in 1845. Mathematical subtleties, however, plagued the subject until the 1950s. The Millikan oil drop experiment relied on the knowledge of the Stokes drag force on the drop due to the viscosity of air.

Specifically, we want to find the solutions for a *constant* and *low-Reynolds* number flow of a *viscous* and *incompressible* fluid past a sphere of radius a. Assume the flow far away from the sphere is along the z-axis, $\vec{v} = V\hat{z}$, and neglect gravity to start with.

(a) For this type of flow, **show** that the fluid equations lead to two Laplace equations for pressure P and vorticity $\vec{\omega} \equiv \nabla \times \vec{v}$:

$$\nabla^2 P = 0\,, \qquad \nabla^2 \vec{\omega} = 0\,. \tag{1}$$

In spherical coordinates where the origin is at the center of the sphere, the boundary conditions of the fluid velocity are (i) $\vec{v} = 0$ at the surface of the sphere, $r = a$; and (ii) $\vec{v} \to V\hat{z} = V(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$ as $r \to \infty$. Now we would like to solve for the velocity at all r. A general solution can be written as: $\vec{v} = V[A(r)\cos\theta\,\hat{r} - B(r)\sin\theta\,\hat{\theta}]$.

(b) **Show** that the incompressibility condition requires

$$B = \frac{1}{2r}\frac{d(r^2 A)}{dr}\,. \tag{2}$$

(c) **Show** that the ϕ-component of the vorticity is

$$\omega_\phi = \frac{V\sin\theta}{r}\left[A - \frac{d^2}{dr^2}\left(\frac{r^2 A}{2}\right)\right]\,. \tag{3}$$

One can then show (but you are not required to do this) that $\nabla^2 \vec{\omega} = 0$ requires

$$A - \frac{d^2}{dr^2}\left(\frac{r^2 A}{2}\right) = \frac{const}{r} \,. \tag{4}$$

The function A must obey the boundary condition $A \to 1$ as $r \to \infty$. Given this condition, a reasonable guess for the solution to eq. (4) is a power law of the form: $A = 1 + b_1/r + b_2/r^2 + b_3/r^3 + ...$

(d) **Show** that A and B must be of the form $A = 1 + b_1/r + b_3/r^3$ and $B = 1 + b_1/2r - b_3/2r^3$. From the boundary condition at the surface of the sphere $r = a$, **show** that the final solution for the fluid velocity is

$$v_r = V\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right), \qquad v_\theta = -V\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right). \tag{5}$$

(e) Now let's solve for the pressure. First, **verify** that

$$P = P_\infty + \frac{\alpha\cos\theta}{r^2} \tag{6}$$

is a solution to the Laplace equation. In fact, this is the only solution that is independent of ϕ and approaches a uniform pressure at infinity (but you don't have to prove this). Determining α is the hardest part of the problem, which can be done from the z-component of the Navier-Stokes equation

$$\partial_z P = \eta \nabla^2 v_z \,. \tag{7}$$

Show that the left-hand side is

$$\frac{\partial P}{\partial z} = \frac{\alpha}{r^3}(1 - 3\cos^2\theta) \,. \tag{8}$$

The right-hand side of eq. (7) can be calculated from eq. (5) after some nasty algebra (you can skip this part):

$$\eta\nabla^2 v_z = \frac{3a\eta V}{2r^3}(3\cos^2\theta - 1) \,. \tag{9}$$

Together, we have $\alpha = -3a\eta V/2$. The final solution for P is therefore

$$P = P_\infty - \frac{3a\eta V\cos\theta}{2r^2} \,. \tag{10}$$

(f) The force (per unit area) exerted by the fluid on the surface of the sphere is related to the stress tensor. It can be shown that $F_r = T_{rr} = -P_\infty + 3\eta V\cos\theta/2a$, and $F_\theta = T_{r\theta} = -3\eta V\sin\theta/2a$. Due to the symmetry of the problem, the net force is along z, the direction of the flow. Using the given F_r and F_θ, **show** that

$$F_z = -P_\infty\cos\theta + \frac{3\eta V}{2a} \,, \tag{11}$$

and that the total force integrated over the sphere's surface is

$$F_{drag} = 6\pi\eta V a \,. \tag{12}$$

This is known as the Stokes' law.

(g) Now let's turn on gravity. For a sphere of mass density ρ_s falling in a viscous fluid of density ρ_f under the influence of gravity, the terminal speed is reached when gravity is balanced by the drag force and the buoyant force. **Show** that the terminal speed is

$$v_{term} = \frac{2ga^2}{9\nu} \left(\frac{\rho_s}{\rho_f} - 1 \right) . \tag{13}$$

Estimate the terminal speed in cm/s for a steel sphere of density ~ 5 g/cm^3 and radius of 1 cm in glycerine at 0 degree Celsius (with $\nu \sim 100$ cm^2/s and density of 1.26 g/cm^3).