Astro/Phys 202: Astrophysical Fluid Dynamics — November 27 2024

C.–P. Ma

Problem Set 9

Due 6pm Thursday December 12

0. **Reading**: Balbus' MHD notes: Chapters 2, 4 and 8; MRI review paper

1. **Ohm's Law**

The familiar version of the Ohm's law, $\vec{J} = \sigma \vec{E}$, is valid only in the frame of the conducting material. In a frame that moves with a constant velocity $\vec{v}$ (for $v \ll c$) relative to the conductor, show that the more general version of Ohm's law is:

$$\vec{J} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) . \tag{1}$$

2. **Magnetorotational Instability**

We discussed in class the dispersion relation for waves traveling with $e^{i(kz-\omega t)}$ in a perfectly conducting rotating fluid in a magnetic field along the z-axis.

(a) Using the dispersion relation and assuming $d\Omega^2/dR < 0$ (almost all astrophysical disks obey this condition), show that magnetorotational instability (MRI) occurs for wavenumbers satisfying

$$(kv_A)^2 < -\frac{d(\Omega^2)}{d\ln R} . \tag{2}$$

(b) Show that the wavenumber of the maximum growth rate is given by

$$(k_{\max} v_A)^2 = -\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2} \right) \frac{d\Omega^2}{d\ln R} , \tag{3}$$

where κ is the epicyclic frequency of the rotating disk.

(c) Show that the maximum growth rate is given by $e^{|\omega_{\max}|t}$, where

$$|\omega_{\max}| = -\frac{1}{2} \frac{d\Omega}{d\ln R} , \tag{4}$$

and $|\omega_{\max}| = 0.75\Omega$ for a Keplerian disk.

3. **Turbulent Eddy Sizes**

The Kolmogorov spectrum for an isotropic turbulent flow is valid only in the range $l_{\min} < l < l_{\max}$. Over this range, the turbulent eddies continuously receive energy from larger length scales and transfer it to smaller scales at a mean rate $\mathcal{E}$. The smallest scales in a turbulent flow, $l_{\rm min}$, is referred to as the Kolmogorov microscale and is the scale below which viscosity ν matters. Using dimensional analysis, show:

(a) The Kolmogorov scale is given by

$$l_{\rm min} \sim \left(\frac{\nu^3}{\mathcal{E}}\right)^{1/4} . \tag{5}$$

(b) The largest eddies have a turnover speed of

$$v \sim (\mathcal{E} l_{\rm max})^{1/3} . \tag{6}$$

(c) The ratio of the largest to smallest eddies is

$$\frac{l_{\rm max}}{l_{\rm min}} \sim \left(\frac{v l_{\rm max}}{\nu}\right)^{3/4} = \mathcal{R}^{3/4} , \tag{7}$$

where $\mathcal{R}$ is the Reynolds number for the flow's largest eddies.

(d) The atmosphere of Venus is made of carbon dioxide with a kinematic viscosity $\nu = 0.07(T/100K)^2$ cm^2/s. Solar energy is added to Venus' atmosphere at a rate $\mathcal{E} \sim 0.1$ erg/s/g. Estimate (i) the size of the smallest eddies; (ii) the turnover speed of the largest eddies; and (iii) the ratio $l_{\rm max}/l_{\rm min}$.