C228: Extragalactic Astronomy and Cosmology — February 7 2025

C.–P. Ma

Problem Set 2

Due 6pm Friday February 14

Reading: Sec 22.1 and 22.2 of Big Bang Cosmology from Review of Particle Physics. Note two differences in conventions: (1) we use dimensionless a and they use dimensional R for the scale factor; (2) our cosmological constant Λ is included in the energy density ρ in the Friedmann equation while they write it out as a separate term in their (22.8).

1. The Fate of Three Universes

Depending on the average energy density, a universe can be open, flat, or closed. Consider three universes with $\Omega_0 = 0.27$, 1.0, and 2.7, respectively. Assume $H_0 = 70$ km s^{-1} Mpc^{-1} and a matter-dominated universe (i.e. ignore radiation and any dark energy component).

(a) Express H_0 in units of inverse years. (Show your work.)

(b) For the closed universe, calculate the time (in years) when the expansion reaches a maximum, and the time when the "Big Crunch" occurs.

(c) For the flat universe, compute $\dot{a}$ in the limit as $t \to \infty$.

(d) For the open universe, use the parametric expression derived in lecture to compute $\dot{a}$ in the limit as $t \to \infty$. Compare your answer to part (c).

(e) Using the fact that $a(t_0) = 1$, compute t_0 for all three models.

(f) On a single plot, draw three curves for the scale factor a vs. proper time t for the three universes. Use linear scales for both axes.

Line up the $a(t)$ curves so that $t_0 = 0$ for **all three models** (i.e. we are shifting the time axis for each model so that all three universes have $a = 1$ at $t = 0$ instead of $a = 0$ at $t = 0$). Your time axis should go from -20 Gyr to +50 Gyr.

Be sure to label the time when the Big Bang occurs on each curve (this will occur in the past at negative t), and the times when the closed universe reaches a maximum and undergoes the Big Crunch.

2. The Robertson-Walker Metric

An alternative form of the Robertson-Walker metric is given by

$$ds^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{1}$$

where χ is a new radial coordinate, and k is the curvature constant.

(a) Find the function $S_k(\chi)$ for each of the three cases of k: $k = 0$, $k > 0$, and $k < 0$.

(b) Using this metric, derive an expression for the ratio of circumference to radius for a circle for each of the three cases of k.

(c) When the radial coordinate χ is small compared with the radius of curvature $|k|^{-1/2}$, what value does the ratio in (b) approach for $k > 0$? What about $k < 0$?

(d) For $\chi = 0.5\,|k|^{-1/2}$, compare the ratio in (b) for a negatively curved space with that of a flat space. Repeat it for a positively curved space.

(e) For $\chi = 15\,|k|^{-1/2}$, compare the ratio in (b) for a negatively curved space with that of a flat space. Comment on any trend you have learned from (c), (d) and (e).