Astro 7B – Problem Set 3

1 Tides and Hot Jupiters

Take Jupiter to be on a circular orbit at a distance r from the Sun. For this entire problem, you may work (for simplicity) in the frame centered on the Sun (don't bother with center-of-mass corrections). The mass of Jupiter is M_J and the mass of the Sun is $M_\odot$.

Now imagine Jupiter is suddenly perturbed — say from the gravitational pull of another planet that happens to pass by in a "hit-and-run" encounter — so that Jupiter's velocity is suddenly reduced by a factor of $f = 1/2$ (but still points in the same instantaneous direction). Assume further that the encounter does not change the planet's instantaneous position. This last assumption is called the "*impulse approximation*": the encounter is assumed to change the instantaneous velocity but not the instantaneous position. An *impulse* is suddenly imparted to Jupiter so that its momentum is instantly changed without changing its position.

(a) **Compute the new orbital semimajor axis a' and the new orbital eccentricity e' after the encounter, in terms of the given r. Your answer should be symbolic (do not plug in for r, but do plug in for f).**

(b) **In the limit that $f \to 0$, what are a' and e'?**

(c) The new orbit's periastron distance, q', is so close to the Sun that large tides are raised on Jupiter by the Sun. Dissipation of this eccentricity tide causes the orbit to change again. **Compute the final semimajor axis a'' and final eccentricity e'' after tidal dissipation has run its course. Express in terms of r.**

ALSO, DRAW A PICTURE, TO SCALE, of the original circular orbit, the perturbed "primed" orbit, and the tidally evolved "double-primed" orbit.

(d) This part is similar to (c), but works in the asymptotic limit that $e' \to 1$. **First express orbital angular momentum L' in terms of q' and e'. In the limit that $e' \to 1$, compute a'' and e'', in terms of q'.** These are handy expressions to have around.

(e) **Write down an expression for the orbital energy lost due to tidal dissipation, $|E_{\rm diss}|$, in terms of a' and a'', the given masses, and fundamental constants. Now simplify your expression for the case $a'' \ll a'$, as would be the case if the orbit prior to tidal dissipation were highly elliptical.** (We take the absolute value since we have defined $E_{\rm diss}$ to be the energy *lost*, and we don't want to have to futz with signs if we don't have to. Energy is definitely lost, as a result of friction.)

(f) Consider a "hot Jupiter" of mass M_J on a circular orbit of radius 0.05 AU around a star of mass $M_\odot$. Such hot Jupiters ("hot" because they are so close to their parent stars) exist in about 0.5% of planetary systems in the universe.

The scenario of circular $\rightarrow$ primed $\rightarrow$ double-primed outlined in this problem has been proposed to explain the existence of hot Jupiters. In this picture, hot Jupiters are supposed to have formed like Jupiter did at $r \approx 5$ AU, but then to have evolved onto their tight orbits as a result of gravitational perturbations + tidal dissipation.

There is a problem (actually more than one problem) with this scenario.[1] **Estimate $|E_{\rm diss}|$, the energy dissipated in tides, in either Joules or ergs.**

Now compare this energy to the gravitational binding energy of the planet. Take the planet to have the same mass and radius of Jupiter, and approximate the planet as being of uniform density. **Estimate $|E_{\rm bind}|$, the magnitude of the self-gravitational binding energy of the planet, in either Joules or ergs.** (Again, we take the absolute value so that we don't have to futz with signs.)

Compare $|E_{\rm diss}|$ to $|E_{\rm bind}|$ and comment.

2 Accretion

Although the answers for this problem will not depend sensitively on factors of 2, try to keep track of such factors anyway, because the factors of 2 for this problem are physically meaningful, as we discussed in lecture.

(a) A white dwarf in a mass-transfer binary is observed to have a total accretion luminosity of $L = 1L_\odot$. The white dwarf has radius $R = 1R_\oplus$ and mass $M = 1M_\odot$. The accretion luminosity is emitted by the accretion disk AND by the boundary layer right near the surface of the white dwarf.

Calculate the white dwarf's accretion rate $\dot{M}$ in units of $M_\odot$/yr.

(b) Assume the accretion luminosity from the surface boundary layer is emitted as a blackbody spectrum and is emitted isotropically from a sphere of radius R. **Deduce from this assumption, and the numbers given in (a), whether the accreting white dwarf radiates primarily at radio, infrared, UV, or X-ray wavelengths (choose one).**

(c) A neutron star in a mass-transfer binary is observed to have an accretion luminosity of $L = 1300L_\odot$. The neutron star has radius $R = 10$ km and mass $M = 2M_\odot$. The accretion

[1] Although it is a problem for hot Jupiters, it is not a problem in other contexts. For example, stars can have their orbits tightened this way in globular clusters.

luminosity is emitted by the accretion disk AND by the boundary layer right near the surface of the neutron star.

Calculate the neutron star's accretion rate $\dot{M}$ in units of $M_\odot$/yr.

(d) Assume the accretion luminosity from the surface boundary layer is emitted as a blackbody spectrum and is emitted isotropically from a sphere of radius R. **Deduce from this assumption, and the numbers given in (c), whether the accreting neutron star radiates primarily at radio, infrared, UV, or X-ray wavelengths (choose one).**

3 Accretion Disk Spectrum

The spectrum of an accretion disk scales as $F_\nu \propto \nu^{1/3}$. Here F_ν is the flux per unit frequency interval (i.e., the flux between frequency ν and $\nu + d\nu$). This problem shows how this scaling can be derived.

Consider a wide spectrum that extends from some low frequency to some high frequency. This spectrum arises from a wide disk that extends from some large radius to some small radius. The large radii are cooler and contribute most to emission at low frequencies. The small radii are hotter and contribute most to emission at high frequencies.

As assumed in class, take the disk to radiate like a blackbody (actually a blackbody having a range of temperatures; a "multi-color blackbody"). Mass flows steadily through the disk at a constant rate $\dot{M}$ and the central mass is M (which is assumed to be constant for this problem).

Finally assume the disk is viewed by the observer face-on at a distance d.

(a) **Write down a formula for the temperature T of the disk as a function of disk radius a in terms of the variables given.** The formula should be exact in the scalings, but the coefficients don't have to be accurate.

(b) Consider just an intermediate annulus of the disk, from radius a to $a' = 2a$. **Derive an approximate expression for the flux F from this annulus, as observed by someone a distance d away. Express in terms of a, M, $\dot{M}$, and d.** You should perform an integral over disk radius. Also assume that the annulus radiates uniformly over the 4π steradians of the celestial sphere centered on the annulus—i.e., assume that the annulus radiates spherically symmetrically. It doesn't in real life, but it's an OK approximation in this context because we are not interested in exact coefficients; we are only interested in scalings.

(c) The inner edge of the annulus is at temperature T — and thus radiates primarily at a frequency $\nu \propto T$ (by the Wien peak law). Likewise, the outer edge of the annulus is at temperature T'—and thus radiates primarily at a frequency $\nu' \propto T'$. The annulus is thus responsible for the spectrum between ν and ν'.

In general,

$$F = \int_{\nu}^{\nu'} F_\nu \, d\nu \,. \tag{1}$$

As long as ν and ν' are not orders-of-magnitude different, we can approximate F_ν as constant. **Make this approximation and write down F in terms of F_ν and ν (rewrite ν' in terms of ν, using the fact that $a' = 2a$).**

(d) **Write down proportionalities for a in terms of T, and T in terms of ν. Thus rewrite your answer in (b) to decide how F scales with ν. Just a proportionality is sufficient. Combine with (c) to decide how F_ν scales with ν. Voilà!**