Astrophysical Fluid Dynamics – Problem Set 10

Readings: Shu pages 372–376 on magnetospheres; Shu pages 98–101 on rotational instability and Rayleigh's criterion; Shapiro & Teukolsky's "Black Holes, White Dwarfs, and Neutron Stars", Chapter 15 on magnetic accretion; Binney & Tremaine's "Galactic Dynamics" on radial epicyclic frequency

Problem 1. Magnetospheres

(a) [5 points] The solar wind blows across the Earth but is largely deflected away by the Earth's magnetosphere. Estimate the radius r_A of the Earth's magnetosphere, in units of the Earth's radius. This is the radius where the Earth's magnetic pressure balances the pressure of the solar wind. Use whatever you need to from class, including previous problem sets. The surface field of the Earth is about 0.5 G (this is NOT the field strength at the magnetospheric boundary) and you may model the Earth's field as a dipole.

(b) [7 points] Consider now a different situation also involving a magnetosphere: an accretion disk orbiting a magnetized star (as discussed in lecture). The accretion disk is truncated at its inner edge by the star's (closed) dipole field. Material from the disk diffuses (somehow, via instabilities of some kind) onto the magnetic field lines and is funneled onto the magnetic poles of the star. Our goal here is to derive a rough formula for the radius r_A of the magnetosphere for this accretion disk case. This problem was drawn from Shapiro & Teukolsky's excellent textbook, "Black Holes, White Dwarfs, and Neutron Stars".

Figure 1 shows that the magnetospheric boundary is actually a fuzzy one, having some radial thickness δ over which flow variables like density, velocity, and magnetic field change radially. Figure 1 also shows that the disk distorts the stellar magnetic field over a vertical length scale of order the disk vertical thickness, H — specifically, the disk creates toroidal field (B_ϕ) from a purely poloidal (and mostly vertical B_z) stellar field, as a consequence of flux freezing; the rotating disk tries to "pull" field lines into the azimuthal direction. For disks, $H \ll r$, where r is the disk radius.

Assuming steady-state and axisymmetry, use the azimuthal $\hat{\phi}$-component of the momentum equation to derive the following order-of-magnitude relation:

$$\frac{\dot{M} v_\phi}{r\delta} \sim B_z^2 \tag{1}$$

valid near the disk midplane at the magnetospheric boundary. Here v_ϕ is the disk gas velocity in the azimuthal direction, $\dot{M} \sim 2\pi\rho H v_r r$ is the disk accretion rate (mass per time crossing a circle of radius r; if this statement is not clear to you, try showing it to yourself), ρ is the gas density, and v_r is the gas radial velocity.

It is not obvious what δ should be, but there are two end-member cases we can consider: either $\delta \sim r$ (the largest relevant length scale) or $\delta \sim H$ (the smallest relevant length scale; Rayleigh stability supports the statement $\min \delta \sim H$, as you will see in Problem 2 of this set).

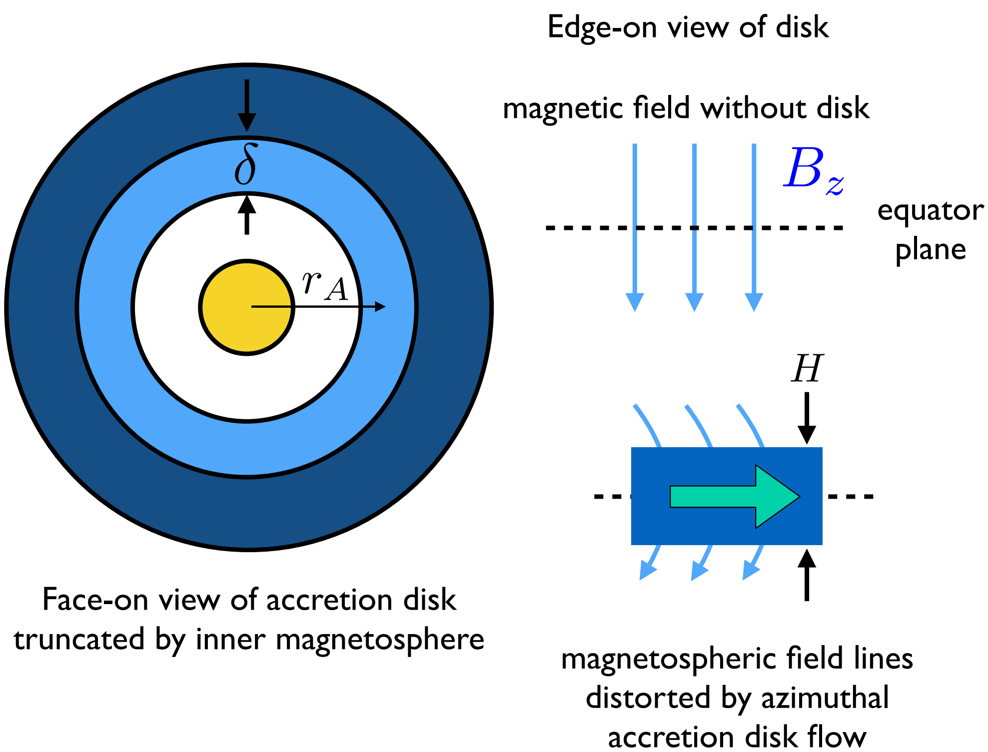

Figure 1: Schematic of magnetospheric truncation of an accretion disk.

To derive (1), assume $|B_r|/\delta \ll |B_z|/H$ and $|B_\phi| \sim |B_z|$ near the magnetospheric boundary. Both assumptions should appear plausible from studying Figure 1.

Suppose $\delta \sim r$: solve (1) for $r = r_A$ as a function of $B_\star$ (the stellar surface field), $R_\star$ (the stellar radius), $M_\star$ (the stellar mass), $\dot{M}$, and fundamental constants. Assume a stellar dipole field. Compare your expression to the one we derived in class for a spherically accreting stellar dipole.

Now suppose $\delta \sim H$: solve for $r = r_A$ as above, now including H as part of your answer.

(c) [3 points] Estimate the radius r_A of the magnetosphere of a young star accreting from a disk, assuming $\delta \sim r$.

Use parameters typical of a T Tauri star: $M_* \sim 1M_\odot$, $R_* \sim 3R_\odot$, a dipole field of surface strength $B_* \sim 1$ kG, and the median measured accretion rate of $\dot{M} \sim 10^{-8} M_\odot\,\mathrm{yr}^{-1}$. Express your answer for r_A in units of AU.

It is known that extrasolar sub-Neptune planets appear less frequently at distances < 0.1 AU from their central stars. This may be because their parent disks were truncated at $\sim$0.1 AU by their host star magnetospheres (no disk, no planet formation). Compare your estimate for r_A to this observed drop-off radius in planet occurrence (see Lee & Chiang 2017).

Problem 2. Radial Epicycles and Rayleigh Stability

Consider an axisymmetric gas disk in 2D. The radial and azimuthal components of the momentum equation read:

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\phi^2}{r} = -\frac{\partial \Phi}{\partial r} - \frac{1}{\rho}\frac{\partial P}{\partial r} \tag{2}$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi u_r}{r} = 0 \tag{3}$$

where u_r is the radial velocity, u_ϕ is the azimuthal velocity, r is the disk radius, P is pressure, ρ is density, and Φ is the gravitational potential (which may include a self-gravitational component but doesn't have to). On the left-hand side we have expanded out all the terms in $(\vec{u} \cdot \nabla)\vec{u}$.

The convective derivative of any scalar x (the derivative following some scalar[1] property x of a fluid parcel) reads:

$$\frac{dx}{dt} = \left(\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + u_\phi \frac{1}{r}\frac{\partial}{\partial \phi}\right) x = \left(\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r}\right) x \tag{4}$$

where the last equality follows because the disk is assumed axisymmetric at all times. Thus we are really tracking how x changes with the radial motion of entire (circular) rings of gas.

[1] I emphasize scalar here because you can also take the convective derivative of a vector, which of course is what is done for, e.g., the inertial term $(\vec{u} \cdot \nabla)\vec{u}$. The terms $-u_\phi^2/r$ and $u_\phi u_r/r$ in (2) and (3) arise from taking the convective derivative of the unit vectors $\hat{r}$ and $\hat{\phi}$—these are non-zero, even for axisymmetric flows.

(a) [3 points] Prove from the ϕ-momentum equation that

$$\frac{d}{dt}(ru_\phi) = 0 \tag{5}$$

The quantity ru_ϕ is the specific angular momentum of gas, which you have just shown is conserved (even as the rings of gas move radially). Call $\ell_z = ru_\phi =$ constant.

(b) [5 points] For the remaining parts of this problem, consider only a single "test ring" (read: test particle) having a strictly constant $\ell_z = ru_\phi$.

Show that the r-momentum equation for the test ring can be written as

$$\frac{du_r}{dt} = -\frac{\partial \Phi_{\text{eff}}}{\partial r} \tag{6}$$

where the effective potential

$$\Phi_{\text{eff}} \equiv \frac{\ell_z^2}{2r^2} + \Phi + h \tag{7}$$

where we emphasize that ℓ_z is a strict constant, and $h = \int(1/\rho)dP$ is the enthalpy (which should be familiar from the Bernoulli constant). The effective potential contains a centrifugal component (sometimes called the "centrifugal barrier"), the gravitational component, and a pressure component.

(c) [5 points] The equilibrium position r_0 of a ring of gas is given by $du_r/dt = 0$. For small-amplitude radial displacements about r_0, we have

$$\begin{aligned} \frac{du_r}{dt} = \ddot{r} &= -\left.\frac{\partial \Phi_{\text{eff}}}{\partial r}\right|_{r_0} - \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2}\right|_{r_0}(r - r_0) && (8) \\ &= -\left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2}\right|_{r_0}(r - r_0) && (9) \end{aligned}$$

which we recognize as the equation for a linear spring with frequency κ given by

$$\kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2} \tag{10}$$

where henceforth it is understood that we are evaluating quantities at r_0 and so drop the $|_{r_0}$ notation. In particular we understand that ℓ_z in Φ_{eff} is specific to r_0 so that $\ell_z = u_{\phi,0} r_0 =$ constant (the specific angular momentum of the ring is conserved while it undergoes radial oscillations).

Neglect for now the enthalpy term in Φ_{eff} (in disks, h is generally small compared to Φ; for now we drop it entirely). Show that

$$\kappa^2 = 4\Omega^2 + r\frac{\partial(\Omega^2)}{\partial r} \tag{11}$$

where $\Omega = u_\phi/r$ is the angular frequency. Show also that

$$\kappa^2 = \frac{1}{r^3}\frac{\partial(\Omega r^2)^2}{\partial r} \tag{12}$$

Thus when $\kappa^2 > 0$—corresponding to flows that are "Rayleigh-stable"—the specific angular momentum Ωr^2 of the background disk increases with increasing radius.

To derive the above relations, recognize from (2) that the equilibrium rotation profile (for zero-pressure disks) is just given by the gravitational potential via

$$\Omega^2 r = \frac{u_\phi^2}{r} = \frac{\partial \Phi}{\partial r} \tag{13}$$

Note this last equation uses Φ, not $\Phi_{\rm eff}$.

(d) [2 points] Astronomers typically assume that $\kappa \sim \Omega$. What is κ for a galactic disk with a flat rotation profile $u_\phi = \Omega r =$ constant, in units of Ω? What is κ for a point-mass (a.k.a. Kepler) potential?

(e) [10 points] Now restore the enthalpy h to consider pressure gradients. Take the gas to have sound speed c_s and recall from PS 1, Problem 3f that the disk's vertical scale height H is given by $H/r = c_s/(\Omega r)$. A disk has $H/r < 1$ (otherwise it wouldn't be called a disk!) which implies the enthalpy $h < \Phi$ (you can check this to order-of-magnitude).

Consider the inner edge of a disk, where the gas pressure decreases (going inward) over a radial length scale δ. Assume a point-mass gravitational potential. Make an order-of-magnitude estimate for the smallest value δ can be before gas at the disk edge becomes Rayleigh-unstable ($\kappa^2 < 0$). Rayleigh stability sets a limit to how sharp disk edges can be (a fact of possible relevance to determining the structure of gaps opened by planets in circumstellar disks).

Hint: the pressure profile $P(r)$ of a smooth annular disk must have a negative second derivative somewhere.