Astrophysical Fluid Dynamics – Problem Set 2

Readings: Pringle & King 1.3–1.4, 1.6.1, 1.7.1, 1.7.3; Shu pages 20–24, 30–33, 49–50; Tritton 10.4–10.7; Landau and Lifshitz Chapter 1, section 5, 6; Movies: e.g., Eulerian/Lagrangian, Fluid Quantity and Flow, Flow Visualization

Problem 1. The Energy Equation

(a) [10 points] This part derives an equation for the total energy density.

Define the total kinetic energy density

$$\frac{1}{2}\rho v^2 = \frac{1}{2}\rho(u^2 + w^2 + 2\vec{u}\cdot\vec{w}) = \frac{1}{2}\rho(u^2 + w^2 + 2u_i w_i) \tag{1}$$

where the total velocity

$$\vec{v} = \vec{u} + \vec{w} \tag{2}$$

is the sum of the mean velocity $\vec{u}$ and the random velocity $\vec{w}$.

In conservation form, the equation for the total kinetic energy density reads:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\nabla\cdot\left(\frac{1}{2}\rho v^2\vec{v}\right) - \rho\vec{\nabla}\phi\cdot\vec{v} \tag{3}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{1}{2}\rho w^2 + \rho u_i w_i\right) &= -\frac{\partial}{\partial x_k}\left[\frac{1}{2}\rho\left(u^2 + w^2 + 2u_i w_i\right)(u_k + w_k)\right] \\ &\quad -\rho\frac{\partial\phi}{\partial x_k}v_k\end{aligned} \tag{4}$$

where we have allowed for a body force density due to gravity ($\vec{f}_{\rm grav} = -\rho\vec{\nabla}\phi$, where ϕ is the gravitational potential).

Take the spatial average $\langle ... \rangle$ of the above equation to derive the *total energy equation* (Shu equation 2.32):

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}u^2 + \rho\varepsilon\right) + \frac{\partial}{\partial x_k}\left[\frac{\rho}{2}u^2 u_k + u_i\left(P\delta_{ik} - \tau_{ik}\right) + \rho\varepsilon u_k + F_k\right] = -\rho u_k\frac{\partial\phi}{\partial x_k} \tag{5}$$

where the internal energy density $\rho\varepsilon \equiv (1/2)\rho w^2$, P is the pressure, δ_{ik} is the Kronecker delta, τ_{ik} is the viscous stress tensor, and $F_k \equiv (1/2)\rho\langle w^2 w_k\rangle$ is the conductive heat flux.

When performing the spatial average, you may assume as we did in class that taking any derivatives commutes with averaging (so $\langle(\partial/\partial t)(...)\rangle = \partial/\partial t\langle ...\rangle$), and that the density ρ can also be taken out of the average (so $\langle\rho\cdot ...\rangle = \rho\cdot\langle ...\rangle$). Also by definition, $\langle\vec{u}\rangle = \vec{u}$, $\langle\vec{w}\rangle = 0$, $\langle w^2\rangle = w^2$, and $\vec{u}$ and $\vec{v}$ are assumed uncorrelated (and for uncorrelated variables, the average of a product is the product of the averages).

(b) [15 points] This part derives an equation for just the internal energy density (subtracting off the bulk kinetic energy density from the total).

As we wrote down in class, the spatial average of the equation for the total momentum density in conservation form reads

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k + P\delta_{ik} - \tau_{ik}) = -\rho\frac{\partial \phi}{\partial x_i} \tag{6}$$

Use this momentum equation, and the equation for the total energy density as derived above in part (a), to derive the *internal energy equation* (Shu equation 2.36):

$$\rho\frac{D\varepsilon}{Dt} = -P\nabla\cdot\vec{u} - \nabla\cdot\vec{F} + \tau_{ik}\frac{\partial u_i}{\partial x_k} \tag{7}$$

where $\vec{F}$ is the conductive flux. Notice the body force of gravity does not appear in the internal energy equation, which makes sense because gravity changes only the bulk energy not the internal energy.

Hint: you may start by multiplying the momentum equation by u_i. You may also need continuity.

Problem 2. Torricelli's Tank or, "A Hole in My Bucket"

A tall cylindrical tank having radius R_1 is filled with water. The top of the tank is open to the air. A nozzle of radius $R_2 \ll R_1$ is opened at the bottom of the tank, and water comes pouring out. Take the height of water in the tank to be h at some time.

(a) [5 points] Derive an expression for the velocity with which water emerges from the nozzle, assuming the flow is steady and inviscid.

(b) [5 points] The assumption of steady flow is not justified over arbitrarily long times, since the water level is continuously dropping. Estimate the timescale over which the flow can be approximated as steady. That is, derive a rough, order-of-magnitude, analytic expression for the timescale over which the flow velocity at a given position changes by no more than, say, a factor of 2.