Astrophysical Fluid Dynamics – Problem Set 6

Readings: Shu 210–217, 230–240; however much of Blandford and Thorne Chapter 17 you like

Problem 1. Post-Shock is Sub-Sonic

Shu Problem 4a, pages 449–450. This problem proves that for any shock, the post-shock fluid flows subsonically with respect to the post-shock sound speed.

[15 points] We re-print here the problem:

Given the jump conditions for plane-parallel shocks,

$$\rho_2 u_2 = \rho_1 u_1 \tag{1}$$
$$P_2 + \rho_2 u_2^2 = P_1 + \rho_1 u_1^2 \tag{2}$$
$$h_2 + u_2^2/2 = h_1 + u_1^2/2 \tag{3}$$

where $h = \varepsilon + P/\rho = \gamma P/[(\gamma-1)\rho]$ is the specific enthalpy of a perfect gas, derive the Prandtl-Meyer relation,

$$u_1 u_2 = c_*^2 \tag{4}$$

where c_*^2 is given by

$$\frac{\gamma+1}{\gamma-1}\frac{c_*^2}{2} = h + \frac{u^2}{2} \tag{5}$$

and equals a conserved quantity across the shock.

Hint: Manipulate the jump conditions into the form

$$(u_1 - u_2)\left(\frac{c_*^2}{u_1 u_2} - 1\right) = 0 \tag{6}$$

i.e., for $u_1 \neq u_2$, $u_1 u_2 = c_*^2$.

Prove, moreover, that

$$c_1^2 \equiv \gamma P_1/\rho_1 < c_*^2 < c_2^2 \equiv \gamma P_2/\rho_2 \tag{7}$$

and that the Prandtl-Meyer relation requires, for a compressive shock, the upstream flow to be supersonic, $u_1 > c_1$, and the downstream flow to be subsonic, $u_2 < c_2$.

Problem 2. Blowing Bubbles

A star emits a wind of constant velocity v_w and mass-loss rate $\dot{M}$. The wind impacts the surrounding interstellar medium (ISM) of density ρ_0 and terminates in a shock (a.k.a. the heliopause, or for the non-Solar case, an astrosphere).[1]

[1] See Wood et al. 2002, 2005 for measurements of the astrospheres of other stars.

(a) [5 points] Argue your way to an expression for the radius R of the shock as a function of time t, assuming that all of the mechanical energy of the wind is used to drive the shocked (swept-up) ISM outward. Assume the swept-up ISM dominates the mass of the expanding bubble.

Hint: the mechanical (kinetic) energy deposited by the star into the ISM constantly grows with time. This is unlike a supernova which releases a fixed amount of energy within a short instant.

(b) [5 points] Zooming in on a small portion of the expanding spherical shock, we see that it behaves as a 1D plane-parallel shock, with fresh uncompressed gas entering the shock and compressed gas leaving the shock from behind.

What is the velocity of the flow just behind the shock, evaluated in the star's rest frame (not following the shock)? Answer this question using the relevant jump condition(s) for a strong (Mach $\gg 1$) shock and $\gamma = 5/3$.

How can you reconcile this result with the fact that far behind the shock, where the stellar wind is still blowing, the flow velocity must be outward at the stellar wind speed v_w? Thereby deduce that there are actually two shocks—a "forward" shock whose radius R you have solved for in (a), and a "reverse" shock whose radius $R_1 < R$.[2] In the forward shock, the ISM gets shocked. In the reverse shock, the stellar wind gets shocked.

Is the velocity you have just solved for appropriate for shocked ISM or shocked wind?

Problem 3. Self-Similar Magic

In class we defined the dimensionless functions for pressure, density, and velocity behind a strong shock expanding into a uniform medium of density ρ_1:

$$\mathcal{P}(\xi) \equiv \frac{P(r,t)}{P_2(t)} = \frac{\gamma+1}{2}\frac{P(r,t)}{\rho_1[u_{\rm sh}(t)]^2} \tag{8}$$

$$\alpha(\xi) \equiv \frac{\rho(r,t)}{\rho_2(t)} = \frac{\gamma-1}{\gamma+1}\frac{\rho(r,t)}{\rho_1} \tag{9}$$

$$v(\xi) \equiv \frac{u(r,t)}{u_2(t)} = \frac{\gamma+1}{2}\frac{u(r,t)}{u_{\rm sh}(t)} \tag{10}$$

where the notation follows that of lecture (note the subtle font difference between the dimensionless $\mathcal{P}$ and the dimensional P). The similarity variable $\xi \equiv r/R(t)$ where $R(t) = A(E/\rho_1)^{1/5}t^{2/5}$ is the radius of the spherical shock front and $u_{\rm sh} = dR/dt$.

The PDEs for mass, momentum, and energy (written down in class, and also expressed in equations 17.42abc of Thorne & Blandford in the Course Reader) can be transformed into a set of more manageable ODEs (equations 17.47abc of Thorne & Blandford):

[2] See the classic papers on this subject by Castor, McCray, and Weaver 1975 and Weaver et al. 1977.

$$0 = 2\alpha v' - (\gamma+1)\xi\alpha' + v(2\alpha' + \frac{4}{\xi}\alpha) \tag{11}$$

$$0 = \alpha v[3(\gamma+1) - 4v'] + 2(\gamma+1)\xi\alpha v' - 2(\gamma-1)\mathcal{P}' \tag{12}$$

$$3 = \left(\frac{2v}{\gamma+1} - \xi\right)\left(\frac{\mathcal{P}'}{\mathcal{P}} - \gamma\frac{\alpha'}{\alpha}\right) \tag{13}$$

where $' = d/d\xi$. The value gained here is that we no longer have two dimensions, r and t, to worry about; we have only a single dimension, ξ, because $R(t)$ "magically" disappeared from the equations (more on magic in part b). The ODEs in this case are still daunting (and coupled and non-linear), but there are straightforward and robust procedures for solving ODEs (e.g. Runge-Kutta and Burlirsch-Stoer).

(a) [10 points] Derive the MOMENTUM ODE (12) from the original momentum PDE.

For optional bonus points, you may derive the other equations for mass (11) and energy (13) (5 bonus points per extra derivation). Equation (13) is probably best derived from $D(P/\rho^\gamma)/Dt = 0$, the form of the energy equation that we discussed in class, where D/Dt is the convective derivative. (We discussed in class how this energy equation is not the same as $P/\rho^\gamma =$ constant, and how the latter is not correct here because different fluid parcels cross the evolving shock front at different times and therefore pick up different entropies $\mathcal{K} = P/\rho^\gamma$.)

The necessary transformations of derivatives like $\partial/\partial r$ and $\partial/\partial t$ were derived in class and are also given in equations (17.45) and (17.46) of Thorne & Blandford, here reproduced for your convenience:

$$\left(\frac{\partial}{\partial t}\right)_r = -\frac{2\xi}{5t}\left(\frac{\partial}{\partial \xi}\right)_R + \frac{2R}{5t}\left(\frac{\partial}{\partial R}\right)_\xi \tag{14}$$

$$\left(\frac{\partial}{\partial r}\right)_t = \frac{1}{R}\left(\frac{\partial}{\partial \xi}\right)_R \tag{15}$$

[[NB: Shu defines the dimensionless functions $v(\xi)$ and $\mathcal{P}(\xi)$ differently than we did in class. He inserts an extra ξ into v and an extra ξ^2 into $\mathcal{P}$. It is not clear to me why he does this; the resulting equations look a bit nicer mathematically, I guess. By comparison, our definitions of v and $\mathcal{P}$ in class are the same as those of Thorne & Blandford (just with different notation). ODEs result from either set of definitions, and the normalizations from either set of definitions are valid (e.g., $\mathcal{P}(\xi = 1) = 1$ and $v(\xi = 1) = 1$ using either set of definitions.]]

(b) [5 points] It may seem a bit magical that $R(t)$ vanishes from the equations, enabling us to transform a set of PDEs into a set of ODEs. Here we ask whether this transformation depended on our initial "inspired" order-of-magnitude scaling that $R(t) \propto t^{2/5}$, which was physically motivated by considering energy conservation.

Suppose more generally that $R \propto t^\beta$ instead (inspired by nothing more than the usual expectation among astronomers that "everything is a power law"). Is the transformation from PDEs into ODEs still possible without explicitly knowing β? What are some implications of your answer?