Astrophysical Fluid Dynamics – Problem Set 7

Readings: All chapters from Sturrock's Plasma Physics reprinted in the Course Reader

Problem 1. $\vec{E} \times \vec{B}$ Drift

Consider uniform electric $\vec{E}$ and magnetic $\vec{B}$ fields each oriented perpendicular to the other.

[10 points] Show that the motion of a charged particle can be decomposed into a fast gyromotion about $\vec{B}$ plus a slow *drift* at velocity

$$\vec{v}_D = c\frac{\vec{E} \times \vec{B}}{B^2} \tag{1}$$

Remarkably, $\vec{v}_D$ is independent of the sign and magnitude of the particle's charge. This is a standard result that can be found in many textbooks (which you are free to look at). Please provide a complete derivation and also sketch the motions of both an electron and a proton, annotating your sketch with directions and magnitudes of motion.

We will see an application of $\vec{E} \times \vec{B}$ drift in the next problem on magnetospheres (Ferraro's Law of Isorotation).

Problem 2. Magnetospheres

The *magnetosphere* is the immediate environment of a spinning, magnetized star, where the magnetic field dominates all other sources of energy density. This problem considers how plasma within the magnetosphere moves, under steady, ideal-MHD conditions.

Work in cylindrical coordinates (d, ϕ, z), where d measures distance from the rotation axis of the star, and in the lab frame where the star is observed to be spinning. Note that at this stage of the problem, the star and its atmosphere/magnetosphere may, in principle, be spinning in a complicated way—the angular frequency $\vec{\Omega} = \Omega\hat{z}$ may vary with position (see part e). This problem will show that $\vec{\Omega}$ is actually constrained by the magnetic field.

(a) [5 points] Assuming infinite electrical conductivity, write down an expression for the electric field $\vec{E}$ seen in the lab frame, in terms of the velocity of the plasma $\vec{v}$ and the magnetic field $\vec{B}$. At this point $\vec{v}$ and $\vec{B}$ are completely arbitrary.

(b) [5 points] Use Faraday's law of induction to prove that $E_\phi = 0$, assuming the electric field is axisymmetric, and that the entire system is in steady state.

(c) [5 points] Combine (a) and (b) to deduce that the plasma's poloidal (non-$\hat{\phi}$) velocity is parallel to the poloidal magnetic field. That is, show that $B_z/B_d = v_z/v_d$.

The poloidal field is the (d, z) component of the field; e.g., in a dipole field, a single poloidal field line traces out a big "ear" on the star.[1] This part (c) shows that gas parcels slide up and down a poloidal field line like beads sliding along a curved wire.

(d) [5 points] Use the ideal MHD induction equation for $\partial\vec{B}/\partial t$, the absence of magnetic monopoles, and the assumption that the *flow* is incompressible to find that

$$\frac{\partial \vec{B}}{\partial t} = (\vec{B} \cdot \nabla)\vec{v} - (\vec{v} \cdot \nabla)\vec{B} \tag{2}$$

(Aside: we saw earlier in the course that subsonic flows were nearly incompressible; in subsonic flows, the ram pressure is less than the thermal pressure. The generalization of that result is that if flow speeds are subsonic and sub-Alfvénic, then flow incompressibility holds; in sub-Alfvénic flows, the ram pressure is less than the magnetic pressure. We will study Alfvén waves later.)

(e) [10 points] Now assume the plasma's poloidal velocity is zero (as would be the case if the plasma is in hydrostatic equilibrium along a poloidal field line). That is, take $\vec{v}$ to be purely azimuthal:

$$\vec{v} = v(\vec{r})\hat{\phi} = \vec{\Omega}(\vec{r}) \times \vec{r} = [\Omega(\vec{r})\hat{z}] \times \vec{r} = [\Omega(\vec{r})d]\ \hat{\phi} \tag{3}$$

where $\vec{r}$ is the displacement vector from the origin (i.e., the radius vector in *spherical* coordinates), and $\Omega\hat{z}$ is the local angular velocity of plasma.

At this stage, there is no reason to believe that Ω cannot vary arbitrarily with position, i.e., the plasma might be differentially rotating in an arbitrary way. In fact, though, part (f) will show that Ω is not free to vary arbitrarily.

Prove from (2) and (3), the condition of steady state, and the absence of magnetic monopoles that

$$\frac{\partial B_d}{\partial \phi} = \frac{\partial B_z}{\partial \phi} = \frac{\partial B_\phi}{\partial \phi} = 0 \tag{4}$$

That is, the field must be axisymmetric.[2] This makes sense: if the stellar magnetic field were non-axisymmetric (e.g., the star had starspots), then the field seen in the lab frame would be non-steady (it would vary with time at a fixed location as a starspot rotated beneath you).

Hint: I proved $\partial B_\phi/\partial\phi = 0$ last, by considering $\partial(\nabla \cdot \vec{B})/\partial\phi$.

[1] Draw a poloidal field line on each limb of the star and you are on your way to drawing Dumbo the elephant.

[2] This is not the same as $\partial\vec{B}/\partial\phi = 0$, since unit vectors in cylindrical coordinates have non-zero ϕ-derivatives.

(f) [5 points] Show that (e) implies $(\vec{B}\cdot\nabla)\vec{\Omega} = 0$. That is, the angular velocity of plasma is constant along a field line. This result is Ferraro's (1937) Law of Iso-Rotation for axisymmetric fields. All the plasma along a given field line rotates rigidly.

Every field line is rooted on the stellar surface. If the stellar surface is rigidly rotating, then Ferraro's Law implies that all the plasma within the entire magnetosphere is rigidly rotating with it. The plasma is not moving on Keplerian orbits.

This picture assumes that the plasma does not unduly load the field, i.e., the thermal and ram pressures of the plasma are negligible compared to the magnetic energy densities. This assumption is violated by the Solar wind, more so beyond the Solar wind's Alfvén point than inside it.

Neutron star magnetospheres are thought to rotate rigidly such that their outer boundaries rotate at nearly the speed of light. Winds from neutron stars can be flung out magnetocentrifugally along open magnetic field lines.

(g) [5 points] Ignore parts (a)–(f) above, and consider a star with a hard surface that is rigidly rotating. The star has a time-steady, axisymmetric, **purely poloidal** magnetic field that extends through all space. Plasma pervades the field, i.e., plasma fills the space above the star.

In the frame rotating with the star, there is only a static magnetic field and no electric field. In this frame, the charged particles of the plasma surrounding the star execute pure gyro-motion around field lines (assuming infinite conductivity, and neglecting forces like gravity, pressure, Coriolis, and centrifugal forces that are assumed small compared to electromagnetic forces; recall lecture where we made this same "EM beats gravity" approximation for the equation of motion for electrons).

How does the plasma move in the lab frame? Show that your answer is consistent with $\vec{E}\times\vec{B}$ drift (problem 1).

Note that parts (a)–(f) did NOT assume the field was purely poloidal. So what we are showing here in part (g) is a weak (restrictive) version of Ferraro's Law of Isorotation. But it is so much easier to see and prove than the more general case!