Order-of-Magnitude Physics – Problem Set 4

Due at the beginning of class.

Do any 1 of the problems + the last question (make up your own question).

You are free to do more if you like; answers will be graded.

Quote of the week:

> "Too much mathematical rigor teaches *rigor mortis*: the fear of making an unjustified leap even when it lands on a correct result. Instead of paralysis, have courage — shoot first and ask questions later." — Sanjoy Mahajan, from *Street-Fighting Mathematics*

Problem 1. Tabletop Dynamos

Magnetic fields $\vec{B}$ in conducting fluids are governed by the induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + D_{\mathrm{M}} \nabla^2 \vec{B} \tag{1}$$

where the first term on the right-hand side governs how magnetic flux is carried along by ("frozen into") fluid parcels, while the second term is the Ohmic dissipation term derived in class. The fluid velocity is $\vec{v}$, time is t, and D_{M} is the magnetic diffusivity.

Magnetic dynamos are fluid systems whose motions ($\vec{v}$) can sustain magnetic fields.

(a) Were fluid motions in the Earth's liquid iron core to cease, how long would it take the Earth's magnetic field to decay?

(b) Why is it so difficult to study dynamos using tabletop experiments?[1]

[1] Despite the difficulty, the Princeton Plasma Physics Lab has performed laboratory experiments related to the magneto-rotational instability, a dynamo mechanism in differentially rotating flows (http://mri.pppl.gov).

Problem 2. Fade to Black

White dwarfs represent collapsed end-states of solar-mass stars that have expired their nuclear fuel. Typically they have masses $M \sim 1M_\odot$ and radii $R \sim 1R_\oplus$. Their interiors are composed of a crystalline lattice of carbon and oxygen nuclei, bathed in a sea of free degenerate electrons: in short, an ultra-dense metal made of carbon and oxygen.

Usually in stars, heat is transported by either convection or radiation. But in white dwarf interiors, heat is transported by electron conduction. Overlying the metallic interior is a thin shell of hydrogen-rich gas — the white dwarf atmosphere — that transports energy by radiation.

Take the core temperature to be $T_{\rm core}$ and the atmospheric temperature to equal $T_{\rm eff}$ (effective temperature of a blackbody). When born, white dwarfs have $T_{\rm eff} \approx 3 \times 10^4$ K (they are UV-bright sources) and $T_{\rm core} \approx 1 \times 10^7$ K. Assume that $T_{\rm core}/T_{\rm eff}$ maintains this ratio of $\sim$300 over the entire cooling history of the white dwarf.

For this problem, neglect corrections to the specific heat that might arise from $T < T_{\rm Debye}$.[2]

Shapiro & Teukolsky, in their classic textbook "Black Holes, Neutron Stars, and White Dwarfs," state that as white dwarfs cool, nearly all of their metallic interiors can be modelled as isothermal. That is, the temperature equals $T_{\rm core}$ nearly everywhere inside the interior. However, they offer no quantitative justification. We try to provide one here.

(a) Estimate the thermal conduction time across the white dwarf interior, as a function of $T_{\rm core}$, assuming the interior is isothermal. Express in yr.

(b) Estimate the thermal cooling time of the white dwarf interior, as a function of $T_{\rm core} (= 300 T_{\rm eff})$, assuming the interior is isothermal. Express in yr. The cooling time is the thermal energy of the dwarf divided by the power (a.k.a. luminosity) it emits into space.

(c) You should find from (a) and (b) that the conduction time should be shorter than the cooling time, especially as $T_{\rm core}$ drops. Given this inequality, show from the diffusion equation that an isothermal interior is a good approximation.

Hint: white dwarfs are spheres.

(d) What are the luminosities of the faintest white dwarfs in the universe today?[3] Express in solar luminosities $L_\odot$.

[2]These corrections turn out to be of order unity; if you look at Shapiro & Teukolsky, there are large numerical coefficients that happen to cancel out any Debye corrections that we might naively make. In this regard, Kittel ("Introduction to Solid-State Physics") points out that the thermal conductivity of metals is roughly constant for $T > T_{\rm Debye}/10 \sim 30$ K but does not explain where the "10" comes from; see the Course Reader, page 28, for the thermal conductivity curve of copper to verify that this is true.

[3]Sometimes referred to as black dwarfs.

Problem 3. Hot Chips

One factor limiting computer chip speeds is the waste heat they produce. The more computations per second a chip performs, the more heat it produces and the greater the danger of electronic failure.

A chip is a collection of transistors and connecting wires. Functionally, a transistor is an on/off switch. Turning the transistor "on" means charging it up — it takes energy to charge it up, just as it takes energy to charge up a capacitor. The faster the CPU speed, the more capacitors are being charged up per unit time (the faster the charges are being shuttled around the chip).

Chip manufacturers are constantly finding new ways to shrink transistors to cram more of them onto a given size chip (Moore's Law). These days a transistor measures $a \sim 300$ Angstroms across.

(a) Using the transistor size a, estimate the waste-heat power output of a modern chip the size of your fingernail. Express in Watts.

(b) The chip heats the air inside the computer casing. A cooling fan is used to flush the hot air out. Use your answer in (a) to estimate how much hotter the air gets inside the computer, in steady state. Express the change in air temperature, ΔT, in K.

Hint: in cgs, capacitance has very nice units.

Problem 4. Ask Your Own Question

Ask an OOM question of your own. You don't have to answer it.