Astro 201 – Radiative Processes – Solution Set 11

by Eugene Chiang

Readings: Article by Kellerman; Rybicki & Lightman 7.1, 7.2, 7.4, as much as you like of 7.6, 7.7; however much of Blandford's lecture notes but watch for errors.

Problem 1. *Compton Catastrophe*

We derived in class that for a synchrotron-self-Compton (SSC) emitting source that is static in bulk, the ratio of the luminosity due to first-generation inverse Compton scattering of synchrotron photons to the luminosity due to synchrotron processes is

$$f \equiv \frac{L_{IC,1}}{L_{sync}} = C\nu_m T_{bm}^5 \tag{1}$$

where ν_m is the frequency at which the synchrotron spectrum peaks, and T_{bm} is the brightness temperature of the synchrotron radiation at that frequency.

(a) Derive an estimate for the constant C in terms of fundamental constants. Assume whatever geometry you like for the source. I found I had to use the (usual) constants e, k, c, and m_e.

You might find the heuristic derivation of the optically thick portion of synchrotron spectra helpful (very indirectly).

We derived in class that $f = U_{rad,sync}/U_B$, where $U_{rad,sync}$ is the energy density of synchrotron radiation, and U_B is the energy density of the background magnetic field.

Also as outlined in class, we obtain a minimum estimate of $U_{rad,sync}$ by taking the energy density near the peak of the synchrotron spectrum (where optically thick transitions to optically thin). We imagine a sphere of synchrotron radiation of radius R, energy density $U_{rad,sync}$, and distance to the observer d. The luminosity of the source is

$$L_{rad,sync} \sim U_{rad,sync} c 4\pi R^2 \tag{2}$$

since $U_{rad,sync}c$ is the internal flux of energy in the source (a flux is a density times a speed; this internal flux is not to be confused with the flux of energy observed by the distant observer), and $4\pi R^2$ is the area of the source.

Now the distant observer measures a flux

$$F_{rad,sync} = L_{rad,sync}/4\pi d^2 \tag{3}$$

so we combine (3) and (2) to write

$$F_{rad,sync} \sim U_{rad,sync} c \frac{R^2}{d^2} \tag{4}$$

But we can also write

$$\begin{aligned} F_{rad,sync} &\sim \nu F_\nu & (5)\\ &\sim \nu I_\nu \frac{\pi R^2}{d^2} & (6)\\ &\sim \nu_m \frac{2kT_{bm}}{\lambda_m^2}\frac{\pi R^2}{d^2} & (7)\end{aligned}$$

where T_{bm} is the maximum brightness temperature of the source measured at wavelength λ_m. Combine (7) and (4) to write

$$U_{rad,sync} \sim \frac{2\pi}{c^3} k T_{bm} \nu_m^3 \tag{8}$$

Now we have to massage the denominator, U_B, into terms of T_{bm} and ν_m. We know that

$$\nu_m \sim \gamma_m^2 \nu_{cyc} \sim \gamma_m^2 \frac{eB}{2\pi m_e c} \tag{9}$$

where γ_m is the Lorentz factor of the electrons that are most responsible for emission at ν_m. We have argued in class that $\gamma_m m_e c^2 \sim 3kT_{bm}$. Putting it all together,

$$\begin{aligned} U_B &= B^2/8\pi & (10)\\ &\sim \frac{\pi m_e^2 c^2}{2e^2}\frac{\nu_m^2}{\gamma_m^4} & (11)\\ &\sim \frac{\pi m_e^6 c^{10}}{162 e^2 k^4}\frac{\nu_m^2}{T_{bm}^4} & (12)\end{aligned}$$

This finishes the massage. Our final answer is therefore

$$f = \frac{324 e^2 k^5}{c^{13} m_e^6} \nu_m T_{bm}^5 \tag{13}$$

(b) If $\nu_m \sim 1\,\mathrm{GHz}$ (as it typically is for compact radio sources), what is the maximum T_{bm} above which $f > 1$? Express in Kelvin.

We solve the last equation in part (a) for T_{bm} given $f = 1$ and $\nu_m = 10^9$ Hz. We get $T_{bm} \sim 1 \times 10^{12}$ K, as advertised in class.

(c) Given f, what is $L_{IC,2}/L_{IC,1}$? That is, what is the ratio of second-generation to first-generation scattered power? Explain your answer.

Well

$$\frac{L_{IC,2}}{L_{IC,1}} = \frac{U_{rad,IC,1}}{U_{rad,sync}} \tag{14}$$

where $U_{rad,IC,1}$ is the energy density of radiation produced by the first-generation of Compton scatterers. But the right-hand-side just equals f. Therefore the answer is f.

(d) Consider the situation after N scatterings, where $N \to \infty$. What is the critical f for which the total luminosity of the source becomes unbounded? Careful estimates will be rewarded.

The total luminosity after N scatterings equals

$$L_{tot} = \sum_{i=0}^{N} L_{IC,i} \tag{15}$$

$$= \sum_{i=0}^{N} f^i L_{sync} \tag{16}$$

$$= L_{sync} \sum_{i=0}^{N} f^i \tag{17}$$

where $L_{IC,0} \equiv L_{sync}$. This sum diverges for $f > f_{crit} = 1$.

Problem 2. *Compton Saturation*

Rybicki & Lightman problem 7.1b (only b. We answered the other parts in lecture.)

See the solution in RL pages 351–352.

Problem 3. *Typical vs. Average (or Lies, Damn Lies, and Statistics)*

Consider a non-relativistic electron moving at speed $v \ll c$. A single photon of original energy ϵ Compton scatters off the electron and suffers an energy shift of $\Delta\epsilon$. Work in the same Thomson limit that we have been working in throughout lecture.

(a) Estimate a typical value for the fractional change in photon energy, $\Delta\epsilon/\epsilon$. Express your answer symbolically. By "typical," we mean any old photon that comes in from any old direction.

From lecture, we wrote down the exact expression for the photon's energy post-scattering:

$$\epsilon_1 = \epsilon_1' \gamma (1 + \frac{v}{c} \cos\theta_1') \tag{18}$$

where the subscript 1 refers to any quantity post-scattering, and the superscript prime refers to any quantity evaluated in the electron's rest frame. The γ here refers to the electron's Lorentz factor (it is very nearly one since the electron is non-relativistic).

Now in the Thomson limit that we have been working in throughout lecture, $\epsilon_1' \ll m_e c^2$, which means that in the electron's rest frame, the collision is elastic: $\epsilon_1' \approx \epsilon'$. Then

$$\epsilon_1 = \epsilon' \gamma (1 + \frac{v}{c} \cos\theta_1') \tag{19}$$

But we know from the rules of Doppler shifting from the lab frame to the electron's rest frame that

$$\epsilon' = \epsilon\gamma (1 - \frac{v}{c} \cos\theta) \tag{20}$$

Substitute this equation into the previous one to find

$$\epsilon_1 = \epsilon\gamma^2 (1 + \frac{v}{c} \cos\theta_1')(1 - \frac{v}{c} \cos\theta) \tag{21}$$

Now in a typical scattering, $\theta_1' \sim 1$ and $\theta \sim 1$ and they are not likely to be the same. Therefore, to order-of-magnitude, replacing $\gamma^2 = 1$, we have

$$\epsilon_1 \sim \epsilon (1 \pm \frac{v}{c}) \tag{22}$$

where the $\pm$ refers to the fact that either the $\cos\theta_1'$ term or the $\cos\theta$ term might win in any scattering; the photon might be upscattered OR downscattered, depending on the geometry of a single collision! Then our final answer reads

$$\frac{\Delta\epsilon}{\epsilon} \sim \pm\frac{v}{c} \tag{23}$$

(b) Consider this same electron flying through a sea of photons all of energy ϵ. Many photons get inverse Compton scattered to and fro.

Averaged over all the photons that got scattered, what is the mean fractional change in photon energy, $\langle\Delta\epsilon/\epsilon\rangle$? You may use our expression for the inverse Compton power scattered by a single electron if you wish. Neglect the change in the electron's velocity as it is bombarded by photons.

This question was answered in class. The easiest way to derive this is to take the net power radiated by a single electron (read: net power imparted to the photons) and divide by the rate of collisions. This gives the mean energy shift for the photons:

$$\langle\Delta\epsilon\rangle = \frac{P_{compton}}{n_{photon}\sigma_T c} \tag{24}$$

$$= \frac{(4/3)U_{photon}\gamma^2\beta^2\sigma_T c}{n_{photon}\sigma_T c} \tag{25}$$

$$= \frac{4}{3}\gamma^2\beta^2\epsilon \tag{26}$$

where the energy density of the photons, U_{photon}, divided by the number density of photons, n_{photon}, equals the mean energy per photon, ϵ. Therefore to leading order in v/c,

$$\frac{\langle\Delta\epsilon\rangle}{\epsilon} = \frac{4v^2}{3c^2} \tag{27}$$

and it is always positive, since we said at the outset that the electron has more energy than the photons at the beginning.

(c) Explain physically why parts (a) and (b) give different answers. (And then read Blandford page 175, if you wish. It is better to think about this problem and achieve your own understanding than to head straight for Blandford right away).

Answer (a) refers to the typical energy shift in a single scattering. A single scattering can either upshift or downshift a photon's energy. To order v/c, the probability of an upshift equals the probability of a downshift. But to order $(v/c)^2$, the probability of an upshift exceeds the probability of a downshift. That is why in part (b), the expectation value of the energy shift is positive and of order $(v/c)^2$. Blandford calls upshifts and downshifts “blueshifts” and “redshifts,” respectively.