Astro 201 – Radiative Processes – Solution Set 12

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Problem 1. *Faraday Rotation*

Consider the propagation of light through a magnetized plasma. The magnetic field is uniform $\vec{B}_0 = B_0\hat{z}$. Light travels parallel to $\hat{z}$.

An electron in the plasma feels a force from the electromagnetic wave, and a force from the externally imposed B-field. Its equation of motion reads

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B}_0 \tag{1}$$

where the electric field $\vec{E}$ can be decomposed into right-circularly-polarized (RCP) and left-circularly-polarized (LCP) waves:

$$\vec{E} = E_0(\hat{x} \mp i\hat{y})e^{i(k_\mp z - \omega t)} \tag{2}$$

where it is understood that the real part should be taken. The upper sign (-) corresponds to RCP waves, while the lower sign (+) corresponds to LCP waves.

In the equation of motion above, we have neglected the Lorentz force from the wave's B-field, since it is small (by v/c) compared to the force from the wave's E-field.

(a) Prove that the solution of the equation of motion reads

$$\vec{v} = \frac{-ie}{m(\omega \pm \omega_{cyc})}\vec{E} \tag{3}$$

where $\omega_{cyc} = eB_0/mc$. This is the hardest part of the derivation for the dispersion relation of RCP and LCP waves. But it is fairly straightforward.

The x-component of the equation of motion (1) reads

$$\dot{v}_x = -\frac{eE_0}{m}e^{i(k_\mp z - \omega t)} - \omega_{cyc}v_y \tag{4}$$

where $\omega_{cyc} = eB_0/mc$, and the y-component reads

$$\dot{v}_y = \pm\frac{ieE_0}{m}e^{i(k_\mp z-\omega t)} + \omega_{cyc}v_x \tag{5}$$

These are coupled first-order ODEs. We take another time derivative of the x-equation, and then plug in the y-equation. We get

$$\ddot{v}_x = \frac{ieE_0}{m}e^{i(k_\mp z-\omega t)}(\omega \mp \omega_{cyc}) - \omega_{cyc}^2 v_x \tag{6}$$

Our ansatz (read: guess) for the solution reads

$$v_x = Ae^{i(k_\mp z-\omega t)} \tag{7}$$

which we insert into (6) to solve for A. We obtain

$$\begin{aligned} A &= \frac{ieE_0}{m(\omega_{cyc}+\omega)}\frac{(\omega \mp \omega_{cyc})}{(\omega_{cyc}-\omega)} && (8)\\ &= \frac{-ieE_0}{m(\omega \pm \omega_{cyc})} && (9) \end{aligned}$$

Repeat the above analysis for v_y. We get

$$v_y = Be^{i(k_\mp z-\omega t)} \tag{10}$$

where

$$B = \frac{-eE_0}{m(\omega_{cyc} \pm \omega)} \tag{11}$$

It is straightforward to finish the proof from here.

(b) Rybicki & Lightman Problem 8.3

See RL's solution on page 357.

Problem 2. *Combined Scattering and Absorption*

Rybicki & Lightman Problem 1.10

See solutions in RL.

Problem 3. *Greenhouse Warming*

Taken from Chamberlain & Hunten Problem 1.3

Assume that solar radiation is absorbed only at the Earth's surface where the albedo is 40%. The re-radiated energy is absorbed mainly by water vapor, which we approximate as a gray absorber with a density scale height of 2 km and total optical depth $\tau = 2$. Plot the temperature distribution with height for radiative equilibrium. What is the temperature discontinuity at the ground? What is the gradient in the air temperature near the ground, in K/km?

First calculate the effective temperature $T_{\rm eff}$, defined in terms of the absorbed flux:

$$(1 - A) \times \frac{L_\odot}{4\pi a^2}\pi R_\oplus^2 = \sigma T_{\rm eff}^4 \times 4\pi R_\oplus^2 \tag{12}$$

where we have assumed that the atmosphere/earth is spherically symmetric. (You could, alternatively, solve this problem just for a patch on the Earth at a certain latitude, longitude, and time, in which case you would have to calculate the local angle of insolation.) Keep in mind, the effective temperature is not an actual physical temperature; it is merely shorthand for the radiative flux.

For $a = 1$ AU and $A = 0.40$, the above equation yields $T_{\rm eff} = 247$ K.

We know from lecture/reading that the temperature at infinity for an atmosphere in radiative equilibrium is $T_0 = T_{\rm eff}/2^{0.25} = 208$ K.

The temperature profile according to the Eddington solution is

$$T = T_0(1 + 3\tau/2)^{1/4} = 208(1 + 3\tau/2)^{1/4}\,\text{K} \tag{13}$$

So at the surface where $\tau = 2$, the *air* temperature is $T_1 = 294$ K, which sounds about right (21 degrees Celsius).

Re: the gradient in air temperature:

$$\frac{dT}{dz} = \frac{dT}{d\tau}\frac{d\tau}{dz} = \frac{3T}{8(1 + 3\tau/2)}\frac{d\tau}{dz} \tag{14}$$

We are told that the density, and therefore the optical depth, falls with a scale height of $h = 2$ km. So $\tau = \tau_1 \exp(-z/h)$ where $\tau_1 = 2$, and so $d\tau/dz = -\tau/h$. Plugging in,

$$\frac{dT}{dz} = -\frac{3T}{8(1+3\tau/2)}\frac{\tau}{h} \tag{15}$$

which at the surface gives $dT/dz = -27.6$ K/km.

Re: the temperature discontinuity with the ground. From lecture/reading, we know that under conditions of M.R.E. (monochromatic radiative equilibrium) $B_\nu(T_g) = B_\nu(T_1) + F_\nu/2\pi$, where F_ν is the radiative flux (constant with height). We generalize this statement for B.R.E. (bolometric radiative equilibrium), and write

$$\sigma T_g^4 = \sigma T_1^4 + F/2 = \sigma T_1^4 + \sigma T_{\text{eff}}^4/2 \tag{16}$$

from which it follows that $T_g = 311$ K. So the temperature discontinuity is $T_g - T_1 = 17$ K.