Astro 201 – Radiative Processes – Solution Set 1

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Readings: Rybicki & Lightman Chapter 1 except the last section 1.8.

Problem 1. *Blackbody Flux*

Derive the blackbody flux formula

$$F = \sigma T^4 \,, \tag{1}$$

from the Planck function for specific intensity,

$$B_\nu = \frac{2h\nu^3}{c^2(e^{h\nu/kT} - 1)} \,. \tag{2}$$

You may use the integral $\int_0^\infty \frac{dx}{x^5(e^{1/x}-1)} = \pi^4/15$.

Those who know what they have to do can stop reading here. Those who need a bit of help should read on.

Imagine a perfectly flat, blackbody patch at the center of a sphere of radius R. The patch has tiny surface area dA and is lying flat at $r = 0$ in the $\theta = \pi/2$ plane (in spherical coordinates, where r, θ, ϕ are the radius, polar angle, and azimuth). Only one side of the patch is warm; the other side is at absolute zero. Assume that the warm side radiates as a perfect thermal blackbody.

A thermal blackbody emits radiation isotropically—i.e., with no preference for direction. In other words, imagine each point on the patch beaming out a tiny, uniform, hemispherical dome of rays (like an umbrella). It is a hemispherical dome and not a full sphere because only one side of the patch is warm. Note that the Planck blackbody function has no variable inside it that specifies direction of emission; the direction of emission doesn't matter for purely thermal emission. The only variables that need to be specified for the Planck function are temperature and frequency (wavelength).

Despite the isotropic nature of the emission, because the patch is geometrically flat, has definite area, and emits only on one side of itself, a detector glued to the inside of the sphere at $r = R$ detects different amounts of radiation depending on where it is placed. For example, if the detector is glued on the half of the sphere that can't see the bright side *of the patch, the detector detects nothing. If the detector is glued to the sphere directly above the patch, the detector picks up the most number of photons, because it sees the full face-on area of the patch (namely, dA). If the detector is glued at an angle to the pole, then it sees less than area dA. If it is glued in the plane of the patch, it sees nothing but a infinitesimally thin line segment.*

This problem asks you to place detectors everywhere on the inside of the sphere and sum up the radiation collected; this operation is equivalent to "integrating the Planck function over all solid angles into which the radiation is beamed."

Calculate the total luminosity (in units of energy/time) emitted by the patch of area dA. You must integrate the Planck function over the entire emitting area (which is maximally dA but in general will be less, depending on the viewing geometry), over all solid angles into which the radiation is beamed, and over all frequencies. Think about placing tiny detectors all over the inside of the sphere and asking how much energy/time each of the detectors receives.

Then divide the luminosity by dA to calculate the total flux (in units of energy/time/area) emitted by the patch. You should recover the usual blackbody flux formula, σT^4. By definition, σT^4 is the total amount of energy radiated per time per unit area of a blackbody surface, radiated into all solid angles and over all frequencies.

To derive the flux (r&l 1.43) from the Planck Distribution (r&l 1.51), I noted that flux has units of $erg{\cdot}s^{-1}{\cdot}cm^{-2}$ while the distributions has units of $erg{\cdot}s^{-1}{\cdot}cm^{-2}{\cdot}Hz^{-1}{\cdot}ster^{-1}$. So clearly I need to integrate out the frequency and angular dependence. Only catch is that I also need to account for the detectors' seeing the full blackbody patch directly above the radiating patch (at $\theta = 0$) but seeing an effective vanishing area ($dA = 0$) off to the side ($\theta = \frac{\pi}{2}$). See fig.1 to understand this better. Hence I need to toss in an extra factor of $cos(\theta)$ to get this right. And since one side is cold, I only integrate θ up to ($\theta = \frac{\pi}{2}$).

Let Φ be the frequency integral of the Planck Distribution at a certain temperature. Then:

$$\begin{aligned} \Phi &\equiv \int_0^\infty B_\nu(T)d\nu \\ &= \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1}dx \\ &= \frac{2\pi^4}{15}\frac{k^4}{c^2h^3}T^4 \end{aligned} \tag{3}$$

in units of $\frac{erg}{s{\cdot}cm^2{\cdot}ster}$. So the total Flux is:

$$\begin{aligned} F &= \int B_\nu(T)d\nu cos(\theta)d\Omega \\ &= \Phi \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta cos(\theta)sin(\theta) \\ &= \pi\Phi \end{aligned} \tag{4}$$

in units of $\frac{erg}{s \cdot cm^2}$. Plugging in Eqn 1:

$$F = \sigma T^4 \text{ where } \sigma \equiv \frac{2\pi^5}{15}\frac{k^4}{c^2 h^3} \tag{5}$$

Note that the above result (Eqn 3) is r&l 1.58.

Problem 2. *Flat Disks*

This problem forms the foundation for understanding the spectral energy distributions of circumstellar disks, e.g., those surrounding pre-main-sequence stars and AGN.[1]

Consider a perfectly flat, blackbody disk encircling a blackbody star. The star has radius R_* and effective temperature T_*. The disk begins at a stellocentric radius far from the star, i.e. $r_i = R_*/\epsilon$, where $\epsilon \ll 1$ is a small but finite constant. The disk extends to infinity.

Calculate the temperature of the disk, $T(r)$, as a function of radius, r. Make whatever approximations you deem necessary in light of the fact that $\epsilon \ll 1$. Be sure you get the scaling of T with r correctly. It is less important that you get the numerical coefficient correctly.

This disk is PERFECTLY flat. Do not consider individual particles in the disk.

Also, neglect radial transport of energy. Each annulus is independent of neighboring annuli. This is a fine approximation for thin disks.

Starlight strikes the flat disk not at normal incidence (as Rybicki & Lightman implicitly assume in their derivation leading up to r&l 1.13), but rather at grazing incidence. A quick but crude way to solve this problem proceeds as follows. Imagine that you are a patch of unit area lying in the flat disk. From your perspective, you see the upper half face of the star. Idealize the upper half face of the star as a single point source lying above the disk plane at height $\sim R_*$. We should model this point source as having a luminosity of order 1/2 the luminosity of the total star, because we can only see the upper half face of the star (if the bottom of the disk were not there, we could see the entire half face of the star).

Naively—and incorrectly—we would say that the flux of light from this source, evaluated at distance r, would be $(L_*/2)/4\pi r^2$, where L_* is the total luminosity of the star. But this would be incorrect, because what we really want is the flux of light *passing through the flat patch of the disk*. This is the light that the disk actually absorbs. The amount of energy crossing the flat patch per time equals $(L_*/2)/4\pi r^2$ TIMES the sine of the angle at which rays from the point source strike the patch. This angle is not 90

[1]The problem has the further distinction that for some reason, whenever I teach this course, it is the one problem that drives some students crazy and that they never forget.

degrees (normal incidence), but is rather equal to $\sim R_*/r \ll 1$. Therefore the true flux passing through the patch equals

$$\frac{(L_*/2)}{4\pi r^2}\frac{R_*}{r}\,. \tag{6}$$

Set this absorbed flux equal to the flux emitted by the patch:

$$\frac{(L_*/2)}{4\pi r^2}\frac{R_*}{r} = \sigma T^4\,. \tag{7}$$

Since $L_* = 4\pi R_*^2 \sigma T_*^4$, we have

$$T = \left(\frac{1}{2}\right)^{1/4} T_* \left(\frac{R_*}{r}\right)^{3/4}\,. \tag{8}$$

Note that the index for the radial scaling is 3/4, not 1/2. The temperature decays more steeply with distance for a flat disk than for, say, a system of planets because rays strike the disk at grazing incidence, not at normal incidence.

We can re-do this problem more carefully by integrating over the upper half face of the star rather than by idealizing the upper half face as a single point source. The result of such an integration is to replace the $(1/2)^{1/4}$ numerical coefficient above with $[2/(3\pi)]^{1/4}$.

Problem 3. *Albedos*

Optical astronomers detect rocky bodies in the solar system by virtue of their reflected sunlight. The received flux is proportional to aA, where a is the albedo (reflectivity) of the object in some optical passband, and A is the area of the object. Typically, albedos of minor bodies in planetary systems range from $\sim$0.01 to $\sim$0.7.

The proportionality of the flux to aA presumes that all of the optical light received from an object is from reflected sunlight. Of course, that is not quite true; there is some contamination at optical frequencies from the object's thermal emission (the fact that it is warm).

Give a quantitative explanation as to why this contamination is not worth troubling over. If you wish, you can adopt as your "case study" a Kuiper belt object of optical albedo 0.07 *and heliocentric distance* 40 *AU, observed in the visual (V) passband. Flesh out the problem as you deem appropriate ("make it up as you go along.") You may find tables of the integrated Planck function, found in the old edition of Allen's Astrophysical*

Quantities, useful. Such tables might also be found in the new Allen, but I have not checked.

Assume that the efficiency of emission at the (infrared) wavelengths at which most of the object's thermal power is radiated is one. Of course, its efficiency of emitting at optical wavelengths is assumed to be less than one (by Kirchoff's law).

The KBO emits as much flux as it absorbs. Flux balance reads

$$(1-a)\frac{L_{\odot}}{4\pi r^2} = \sigma T^4 \tag{9}$$

where $(1-a)$ is the fraction of incident light absorbed by the KBO. (Here, unlike in problem 2, we are taking sunlight to strike the KBO at normal incidence.) Plugging in the numbers from the case study, we find that the temperature of the KBO is about

$$T \approx 62\,\mathrm{K}\,. \tag{10}$$

This little calculation ignores all real-world complications such as rotation of the body, latitudinal variation of incident flux, finite thermal conductivity, etc., that professional planetary scientists actually do trouble themselves over (sometimes for good reason, but many times for no good reason, in my opinion). The thermal blackbody spectrum of the KBO peaks at a wavelength of about $\lambda_{peak} \sim hc/3kT \sim 77$ microns, or the far-infrared. There is no need to worry about optical thermal emission from this cold blackbody; at a visible wavelength of $\lambda = 0.5$ micron, the specific intensity from the KBO is

$$I_{\nu,KBO-emission} = B_{\nu}(T=62\mathrm{K}) = \frac{2\mathrm{h}\nu}{\lambda^2(\mathrm{e}^{\mathrm{h}\nu/\mathrm{kT}}-1)} = \frac{2\mathrm{h}\nu}{\lambda^2(\mathrm{e}^{465}-1)}\,, \tag{11}$$

well on the Wien exponential tail of the blackbody. We would like to compare this thermal specific intensity with the reflected specific intensity. The reflected light spectrum of the KBO matches in shape, *but not in magnitude*, the spectrum of the sun, which is a blackbody that peaks in the optical. In other words, the reflected spectrum of the blackbody is a *dilute blackbody*; $I_{\nu,KBO-reflect} = fB_{\nu}(T_{\odot})$, where $f \ll 1$. Use energy conservation to find f. We know that the total reflected luminosity of the KBO must equal

$$aL_{\odot}\frac{\pi R_{KBO}^2}{4\pi r^2}\,, \tag{12}$$

in other words, the albedo-diluted power intercepted from the sun by the KBO, where R_{KBO} is the radius of the KBO. But we also know that the reflected light luminosity can be obtained by integrating $I_{\nu,KBO-reflect}$ over all frequencies (to get $f\sigma T_{\odot}^4$) and then multiplying by the surface area of the KBO. This gives $f\sigma T_{\odot}^4 4\pi R_{KBO}^2$. Set this expression equal to the one above, and use $L_{\odot} = \sigma T_{\odot}^4 4\pi R_{\odot}^2$ to find

$$f = a\frac{R_{\odot}^2}{4r^2} \approx 2 \times 10^{-10}\,. \tag{13}$$

Therefore, the reflected specific intensity from the KBO equals

$$I_{\nu,KBO-reflect} = a\frac{R_{\odot}^2}{4r^2}\frac{2h\nu}{\lambda^2(e^{4.8}-1)} \tag{14}$$

which is greater than $I_{\nu,KBO-emission}$ by a factor of $e^{460}f >> 1$ despite the smallness of f. So at $\lambda = 0.5\mu$m, there is no worry of contamination. Now the visible passband is actual a broadband filter, so technically we need to integrate the power from $\sim$0.4μm to $\sim$0.6μm, but there seems to be little point because e^{460} is a mighty big number that's hard to drive down, even if we go to $\lambda = 0.6\mu$m. The moral of the story: the exponential Wien tail really kills the emission at frequencies higher than the peak frequency.

Problem 4. *Make like a tree and ...*

If all the leaves of a tree fall to the ground, how thick is the layer of leaves on the ground? Give your answer in leaf thicknesses, to order-of-magnitude.

What is the order-of-magnitude of the optical depth presented by the leafy tree to someone lying below the tree? Why does this answer make sense from the perspective of the vegetation? (Be the tree.)

The above two paragraphs are flip sides of the same question. You can try the second paragraph before the first, or vice versa.

For simplicity, I'll consider trees with leaves that bunch into vertical cylinders (see attached fig. 3). Let's say these trees have a height $s = 10m$. Each leaf should have roughly the surface area of my driver's license ($\sigma \approx 3in \times 2in \approx 5in^2 \approx 30cm^2$). Looking outside, I guess that trees have between 10 and 100 $\frac{leaves}{m^3}$. So I'll split the difference and say that there are $5 \times 10^{-5}\frac{leaves}{cm^3}$.

This means that the optical depth (assuming constant density) is: $\tau = n\sigma s = (5 \times 10^{-5})(30)(1000) = 1.5$.

This makes sense. A tree ought to absorb most of the light falling on it (i.e. $\tau \approx 1$). If it absorbed any less, it could grow leaves near the base to absorb the extra light. And if it absorbed more, then leaves near the base would die due to lack of light exposure. In fact τ probably determines the density of leaves.

As for the part about the depth of leaves on the ground, consider a 10 m tall square column of leaves with a cross sectional area of 5 in^2 that all fall off the tree and collect directly on top of each other on the ground. With the same density as guessed above, there should be: $nAh = (50\ \frac{leaves}{m^3})(3 \times 10^{-3} m^2)(10m) = 1.5\ leaves$.

So the pile is between 1 and 2 leaves thick. Gee-wiz! That's the same as the first part! The questions are mathematically identical!