Astro 201 – Radiative Processes – Solution Set 2

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Problem 1. *Flat Disk SED*

We return to the perfectly flat, blackbody disk encircling a blackbody star of problem set 1.

(a) Explain why νF_ν is a measure of the "flux radiated by an object per logarithmic interval in frequency." Here F_ν is the flux density [units of energy per time per area per frequency], and ν is the frequency of radiation. (Some refer more loosely to νF_ν as the flux radiated per octave in frequency. An octave, in either the acoustic or electromagnetic spectrum, represents a factor of 2 in frequency.)

Can you understand why νF_ν, and not F_ν, is a quantity of interest to those who wish to understand the overall energetics of an object? It is called the "broadband spectral energy distribution," or "broadband SED," or "SED" for short.

Hint: Most macroscopic objects in the universe are broadband emitters; that is, they radiate continuum radiation at all wavelengths. Plotting their spectrum would give a curve that varies smoothly with frequency. For any object, you can ask whether it is putting out its energy predominantly in X-rays, gamma rays, radio waves, infrared waves, etc (is it an "X-ray object?" "a gamma-ray object?" "an infrared object?") Now ask yourself why a person asking such questions should plot νF_ν and not F_ν.

An infinitesimal flux dF radiated into a logarithmic frequency interval d$(\ln \nu)$ is given by:

$$\text{SED} \equiv \frac{dF}{d(\ln \nu)} = \frac{dF}{\frac{1}{\nu}d\nu} = \nu \frac{d}{d\nu}\left(\int_0^\infty F_\nu d\nu\right) = \nu F_\nu$$

where we have used $d(\ln x) = dx/x$ and $F \equiv \int_0^\infty F_\nu d\nu$. If we are interested in the energetics of some radiating object, the SED would be the quantity of interest, because the specific flux F_ν has units:

$$[F_\nu] \sim \frac{\text{erg}}{\text{Hz cm}^2\text{ s}}$$

which, since it is divided by frequency (proportional to energy for photons), gives a number density of photons. To recover the energy within a given frequency band, we need to multiply by frequency (which is proportional to energy).

In other words, $\nu F_\nu = dF/d\ln\nu$; it is a measure of the total amount of energy per time per area (dF) over a logarithmic interval of frequency ($d\ln\nu$). It is a crude integral of F_ν over a logarithmic interval in frequency (from ν to $e \times \nu$). Whatever frequency νF_ν peaks for a broadband emitter, that is the frequency where most of the energy is being emitted. If νF_ν peaks in the infrared, then we say the object is emitting most of its energy at infrared wavelengths.

(b) Is νF_ν equal to λF_λ, where λ is the wavelength of the radiation, and F_λ is per wavelength rather than per frequency?

Comparing νF_ν to λF_λ, using $\lambda = c/\nu$:

$$\lambda F_\lambda \equiv \frac{dF}{d(\ln \lambda)} = \lambda \frac{dF}{d\lambda} = \frac{c}{\nu}\frac{dF}{d(c/\nu)} = \frac{c}{\nu}\frac{dF}{(-\frac{c}{\nu^2}d\nu)} = -\nu\frac{dF}{d\nu} = -\nu F_\nu$$

so formally they differ by a minus sign. The minus sign arises because going up in wavelength means going down in frequency. We forget about this minus sign because we know where we're going, and energy in radiation is always positive.

(c) Write down an expression for νF_ν for the blackbody disk. That is, write down the formula for the spectrum of the disk as measured by an observer for whom the disk is a point source.

Recall that the disk has a temperature $T(r)$ at every stellocentric radius, r. The inner radius of the disk is r_i and the outer radius is r_o. The disk is a distance D away from the observer, and is inclined by an angle i ($i = 0$ corresponds to a face-on disk). Your expression should take the form of an integral.

Recall that the disk has temperature $T(r)$ for each stellocentric radius r given by:

$$T(r) = \left(\frac{2}{3\pi}\right)^{\frac{1}{4}} T_* \left(\frac{R_*}{r}\right)^{\frac{3}{4}}$$

where R_* and T_* are the radius and temperature of the blackbody star at the center of the disk. To get the SED of the disk, we only have to add up the blackbody flux seen by a distant observer for each infinitesimal annulus of the disk at temperature $T(r)$ and multiply by the frequency ν. Now the flux seen by a distant observer (to whom the disk appears to be a point source) will be given by flux $\sim$ specific intensity $\times$ solid angle. As seen by the distant observer at distance D, each annulus, which is oriented at an angle i with respect to the observer's line of sight, presents a solid angle of:

$$d\Omega = \frac{dA}{D^2} = \frac{2\pi r dr \cos i}{D^2}.$$

Applying equation (1.3b) of Rybicki and Lightman, we get:

$$\nu F_\nu = \nu \int B_\nu(T(r)) \cos\theta \, d\Omega \quad (1)$$

$$= \nu \int_{r_i}^{r_o} B_\nu(T(r)) \frac{2\pi r}{D^2} \cos i \, dr \quad (2)$$

$$= \frac{4\pi h \nu^4}{c^2} \frac{\cos i}{D^2} \int_{r_i}^{r_o} \frac{r dr}{\exp\left[\left(\frac{3\pi}{2}\right)^{1/4} \frac{h\nu}{kT_*} \left(\frac{r}{R_*}\right)^{3/4}\right] - 1} \quad (3)$$

Here we have assumed $\cos\theta \sim 1$ because the disk is a point source to the observer.

(d) Sketch (no heroics necessary) νF_ν vs. ν. If the spectrum exhibits power-law behavior, give the slope of the power law (i.e., $d\ln(\nu F_\nu)/d\ln\nu$). Overlay on your sketch the SED of the central stellar blackbody. Log-log space is best.

The accompanying .ps or .pdf figure shows the disk's contribution to the total SED. To make this plot, I assumed $i = 0$, $T_* = 4000\,\mathrm{K}$, $R_* = 2.5R_\odot$, $r_i = 6R_*$, and $r_o = 2.3 \times 10^4 R_*$. These are typical T Tauri star parameters.

Indeed the disk SED does fall off like a power-law, but much less steeply than the Rayleigh-Jeans tail of the stellar blackbody. We say that the disk is responsible for an "infrared excess" compared to the stellar photosphere. We can derive the index to the power law of the disk SED by making a change of variable to our integral in part (c). Change to the new variable $x \equiv (h\nu/kT_*)(3\pi/2)^{1/4}(r/R_*)^{3/4}$. Then the denominator in the integral is just $e^x - 1$. The numerator $rdr = \text{constant} \times \nu^{-8/3}x^{5/3}dx$. The constant $\times\nu^{-8/3}$ can be taken outside of the integral. The integral becomes $\int_{x_i}^{x_o}[x^{5/3}/(\exp(x) - 1)]dx$. Now x_i and x_o both depend on ν, but provided we are interested in frequencies far from those characterizing the inner and outer radius, we can take $x_i \approx 0$ and $x_o \approx \infty$. Then the integral is just some dimensionless number that we could look up in an integral table if we wanted to. Therefore, $\nu F_\nu \propto \nu^4 \times \nu^{-8/3} \propto \nu^{4/3}$. The index is therefore 4/3—plenty less steep than the Rayleigh-Jeans tail of the stellar blackbody, which gives an index of 3.

The lecture in class gives an alternative, more physical way of deriving the index of 4/3.

Finally, at the very lowest frequencies, the disk will exhibit Rayleigh-Jeans behavior like any good blackbody: $\nu F_\nu \propto \nu^3$.

Problem 2. *Practice with j_ν, α_ν, S_ν, B_ν, I_ν*

(a) A plane-parallel slab of uniformly dense gas is known to be in LTE (local thermodynamic equilibrium) at a uniform temperature T. Its thickness normal to its surface is s. Its absorption coefficient is $\alpha_{\nu,\mathrm{gas}}$. Write down the specific intensity, I_ν, viewed normal to the slab, in terms of the variables given.

A slab having uniform source function S_ν and total normal optical depth τ_ν has a specific intensity normal to its surface of

$$I_\nu = S_\nu(1 - e^{-\tau_\nu}) \tag{4}$$

This is a result worth remembering. Now $\tau_\nu = \alpha_{\nu,\mathrm{gas}}s$. Furthermore, LTE tells us that $S_\nu = B_\nu$. Then

$$I_\nu = B_\nu(T)(1 - e^{-\alpha_{\nu,\mathrm{gas}} s}) \tag{5}$$

(b) The same slab is now filled uniformly with non-emissive dust having absorption coefficient $\alpha_{\nu,\mathrm{dust}}$. The dust is non-emissive, so its emissivity $j_{\nu,\mathrm{dust}} = 0$. Write down I_ν viewed normal to the slab, in terms of all variables given so far.

The source function now changes. The source function is the TOTAL emissivity of the gas divided by its TOTAL absorption coefficient. The total emissivity is still given by the gas in part (a): $j_\nu = \alpha_{\nu,\mathrm{gas}} B_\nu$. But the total absorption coefficient is now greater because of the extra dust: $\alpha_\nu = \alpha_{\nu,\mathrm{gas}} + \alpha_{\nu,\mathrm{dust}}$. Therefore $S_\nu = \alpha_{\nu,\mathrm{gas}} B_\nu / (\alpha_{\nu,\mathrm{gas}} + \alpha_{\nu,\mathrm{dust}})$. Returning to the equation (4) worth remembering, we find

$$I_\nu = \frac{\alpha_{\nu,\mathrm{gas}} B_\nu}{(\alpha_{\nu,\mathrm{gas}} + \alpha_{\nu,\mathrm{dust}})}(1 - e^{-(\alpha_{\nu,\mathrm{gas}} + \alpha_{\nu,\mathrm{dust}})s}) \tag{6}$$

(c) The slab of gas and dust is further mixed with a third component: an emissive, non-absorptive uniform medium having emissivity $j_{\nu,\mathrm{med}}$ and absorption coefficient $\alpha_{\nu,\mathrm{med}} = 0$. Write down I_ν viewed normal to the slab, in terms of all variables given.[1]

This is the same as part (b) except that now the total emissivity is now greater: $j_\nu = \alpha_{\nu,\mathrm{gas}} B_\nu + j_{\nu,\mathrm{med}}$. Then

$$I_\nu = \frac{(\alpha_{\nu,\mathrm{gas}} B_\nu + j_{\nu,\mathrm{med}})}{(\alpha_{\nu,\mathrm{gas}} + \alpha_{\nu,\mathrm{dust}})}(1 - e^{-(\alpha_{\nu,\mathrm{gas}} + \alpha_{\nu,\mathrm{dust}})s}) \tag{7}$$

Problem 3. *Photon Pinball in Clouds*

This problem shows how the usual $e^{-\tau}$ attenuation factor for flux passing through an absorbing medium can be incorrect for flux passing through a purely scattering medium. (And thank goodness, or else it would be really dark on cloudy days.) Clouds can be viewed as a purely scattering medium.

Idealize the cloud as a uniform, 1-dimensional slab comprising particles that can only scatter light. A uniform flux of photons irradiates the top of the cloud deck (that is, just above the cloud deck, there exists a uniformly bright sheet). A photon passing through the cloud gets bounced like a pinball from cloud droplet to cloud droplet, preserving its frequency and never getting absorbed by any droplet. A few photons are lucky enough to

[1] A physical realization of this problem might be an HII region surrounding an ionizing O star. The material in LTE would be the fully ionized plasma, emitting thermal bremsstrahlung radiation. The dust would be dust. The emissive, non-absorptive medium would be the same ionized plasma emitting recombination (line) radiation. For the assumptions stated in the problem to be valid, we would have to evaluate ν at, say, an optical recombination line like Hα.

make it through the cloud, while most get pinballed back out the way they came. We will calculate the fraction that make it through.

Take the cloud to have a droplet density [droplets per cubic volume] η, the droplet radius to be R, and the vertical thickness of the cloud to be z_{max}. Measure vertical distance through the cloud by z, where the top of the cloud is located at $z = 0$ and the base of the cloud is located at $z = z_{max}$.

(a) Write down the optical depth of the cloud, τ.

Optical depth is given by the formula $\tau =$ (density) $\times$ (cross-section) $\times$ (thickness), which, assuming the droplets have a geometric cross-section, gives:

$$\tau(z) = \eta \pi R^2 z.$$

(b) Incident photons from the sun strike the top of the cloud. The photons have a number flux, F_i [number per time per area]. What is the number density of incident photons at the top of the cloud? Call this photon number density n_i. These incident photons have NOT been scattered yet by any droplet. Remember that flux is a number density multiplied by a speed.

In a time Δt the photons will have moved a distance $c\Delta t$, so the volume they occupy during that time is the area A they are passing through times this distance. The incident flux is given by $F_i = \#/A\Delta t$. The incident number density is therefore given by:

$$n_i = \frac{\#}{\text{vol}} = \frac{\#}{Ac\Delta t} = \frac{F_i}{c}.$$

(c) These photons pinball/random walk through the scattering droplets. Random walks are described by the diffusion equation,

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial z^2} \tag{8}$$

where D is the diffusion coefficient and n is the photon number density.

Express D in terms of symbols defined above and whatever fundamental constants you deem appropriate. Hint: dimensional analysis may prove useful.

By dimensional analysis, the diffusion coefficient must have units $[D] = \frac{\text{length}^2}{\text{time}}$. The relevant quantities for diffusion that give these units are the particle velocity $[c] = \frac{\text{length}}{\text{time}}$ and the mean free path $[\lambda] =$ length. Digging through Rybicki and Lightman (1.31) we find that the mean free path is $\lambda = 1/n\sigma$, so using what we know about the scattering cross-section from (a) we find:

$$D = c\lambda = \frac{c}{\eta \pi R^2}.$$

This diffusivity is *microscopic* quantity. Therefore we cannot the macroscopic lengthscale z_{max}; we must use the microscopic lengthscale λ.

(d) In steady-state, $\partial n/\partial t = 0$ (the number density of photons everywhere in the cloud does not change with time). Write down the solution to the diffusion equation for $n(z)$. You should have two, as yet unknown, constants of integration.

The general solution to $\partial^2 n/\partial z^2 = 0$ is $\boxed{n = Az + B}$, where A and B are constants.

(e) To solve for the two constants of integration, you need two boundary conditions. The first condition is that $n_t = n(z = z_{max})$. Here n_t is the number density of photons at the base of the cloud. These photons comprise the transmitted flux.

The second condition is that the (net, number) flux, F, of photons at $z = 0$ equals the incident flux, F_i (directed down into the cloud) MINUS the outgoing, reflected flux, F_r (directed up, away from the cloud into space). Recall Fick's law, which is just another way of writing the diffusion equation, that the (net) flux $F = -D\partial n/\partial z$. So we have $F(z = 0) = F_i - F_r$.

Use the above, and the fact that the incident flux, F_i, must equal the reflected flux, F_r, PLUS the transmitted flux, F_t, to calculate T, the ratio of the transmitted flux to the incident flux, in terms of τ. Are you glad that clouds scatter but do not absorb light?

At $z = 0$, there are TWO contributions to the local photon number density. The first contribution is from incident photons, n_i. The second contribution is from photons reflected out of the cloud, n_r. Therefore, at $z = 0$,

$$n = n_i + n_r = B \tag{9}$$

Now $F_i = F_r + F_t$ means that $n_i = n_r + n_t$ (just divide by c). Use the latter equation to write $n_r = n_i - n_t$ and substitute into (9) to solve for

$$B = 2n_i - n_t \tag{10}$$

At $z = z_{max}$, $n = n_t = Az_{max} + B$. Use this equation and equation (10) to solve for

$$A = \frac{2(n_t - n_i)}{z_{max}} < 0 \tag{11}$$

Now, the net flux $F = -D\partial n/\partial z = -DA$. Notice the net flux is constant with height, as it must be lest the photon density increase or decrease somewhere in the cloud, violating our steady-state assumption. At $z = z_{max}$, $F = F_t = -DA$. Then $n_t = -DA/c$. Note that since $A < 0$, $n_t > 0$ which is correct (you cannot have a negative number density). Thus,

$$\begin{aligned} T &\equiv n_t/n_i & (12)\\ &= -\frac{DA}{cn_i} & (13)\\ &= -\frac{2D(n_t - n_i)}{z_{max}cn_i} & (14)\\ &= -\frac{2(T-1)}{\tau} & (15) \end{aligned}$$

Finally, solve for T in terms of τ:

$$\boxed{T = \frac{2}{2+\tau}} \quad (16)$$

which is certaintly not exponentially sensitive to optical depth. And thank goodness, or else it would be plenty darker on the Earth on cloudy days. Note our expression has the right limits: the transmission goes to 1 if $\tau = 0$, and goes to zero (linearly) as τ goes to infinity.

This pure scattering transmission coefficient T gives much more light passing through the cloud than a purely absorbing cloud (attenuation $e^{-\tau}$) of the same optical depth. For example, with $\tau = 1$, we have

$$\begin{aligned} T &= 0.67 & (17)\\ e^{-\tau} &= 0.37 & (18) \end{aligned}$$

For $\tau = 2$ we have

$$\begin{aligned} T &= 0.50 & (19)\\ e^{-\tau} &= 0.14 & (20) \end{aligned}$$

Absorbing clouds would make the sky much darker than scattering clouds with the same droplet parameters.

Problem 4. *Galaxies*

In an idealized model of a galaxy, stars are uniformly distributed within a cylinder of radius R and height H. The number density of stars is n [in units of stars per volume]. Model each star as a spherical blackbody of temperature T_* and radius R_*. The density of stars is so low that of all possible lines of sight running through the galaxy, the fraction that intersect a star is $\ll 1$.

An observer resolves the galaxy in a face-on viewing geometry, but does not resolve the individual stars making up the galaxy. In other words, the observer detects the galaxy as a circular disk that is very nearly uniformly bright from center to edge.

Write down the specific intensity the observer measures. Neglect terms of order H/d, where d is the distance to the galaxy.

The answer is $I_\nu = B_\nu(T_*) n\pi R_*^2 H$. An easy way to see this is note that the specific intensity of a single star is B_ν, but that we have to dilute this intensity by the fraction of area covered by stars, which is $n\pi R_*^2 H$ (which is τ).

But we can also answer this question in a formal way, which is worthwhile for those who like to test their grasp of formalisms. The galaxy is an example of our proverbial uniform slab. We know that for uniform slabs, $I_\nu = S_\nu \times (1 - e^{-\tau_\nu})$. For optically thin slabs (which galaxies are), $\tau_\nu \ll 1$ and so $I_\nu \approx S_\nu \tau_\nu$. But we also know $\tau_\nu = n\sigma H = n\pi R_*^2 H$, so $I_\nu \approx S_\nu n\pi R_*^2 H$.

It remains to show that $S_\nu = B_\nu$. Well by definition, $S_\nu = j_\nu/\alpha_\nu$. The extinction coefficient $\alpha_\nu = n\sigma = n\pi R_*^2$. The emissivity $j_\nu = nL_\nu/4\pi$, where the 4π accounts for the per solid angle in j_ν, and L_ν is the "luminosity density of a star", defined such that $\int L_\nu d\nu = L_* = 4\pi R_*^2 \sigma_{SB} T_*^4$. Now it must be that $L_\nu = CB_\nu$ where C is some constant, since all the frequency dependence of L_ν must be captured by B_ν. So we solve for the normalization constant:

$$\int L_\nu d\nu = C \int B_\nu d\nu = C \frac{\sigma_{SB} T_*^4}{\pi} = 4\pi R_*^2 T_*^4$$

which implies that $C = 4\pi^2 R_*^2$. So $j_\nu = nB_\nu \pi R_*^2$, which means that yes, $S_\nu = B_\nu$.