

## Astro 201 – Radiative Processes – Solution Set 3

by Katie Peek and Eugene Chiang

**Problem 1.** *The Eddington Limit (Rybicki & Lightman 1.4)*

(a) Show that the condition that an optically thin cloud of material can be ejected by radiation pressure from a nearby luminous object is that the mass to luminosity ratio ( $M/L$ ) for the object be less than  $\kappa/(4\pi Gc)$ , where  $G =$  gravitational constant,  $c =$  speed of light,  $\kappa =$  mass absorption coefficient of the cloud material (assumed independent of frequency).

Force of radiation must exceed the force of gravity:

$$f_{rad} > f_G, \quad (1)$$

where

$$f_{rad} = \frac{\kappa}{c} \frac{L}{4\pi R^2} \quad (2)$$

and

$$f_G = \frac{GM}{R^2}. \quad (3)$$

Note that  $L/c$  is a “momentum luminosity” (units of momentum per time). The factor of  $\kappa$  is a cross-section for absorbing this momentum (energy), per unit mass. Thus, the condition for the cloud’s ejection is given by

$$\begin{aligned} \frac{\kappa}{c} \frac{L}{4\pi R^2} &> \frac{GM}{R^2} \\ \frac{\kappa}{4\pi Gc} &> \frac{M}{L}. \end{aligned} \quad (4)$$

(b) Calculate the terminal velocity  $v$  attained by such a cloud under radiation and gravitational forces alone, if it starts from rest a distance  $R$  from the object. Show that

$$v^2 = \frac{2GM}{R} \left( \frac{\kappa L}{4\pi GMc} - 1 \right).$$

Initially at rest at position  $R$ , the cloud is simultaneously pulled inward under gravitation and pushed outward by radiation. If the cloud begins moving outward, the force due to radiation exceeds the force due to gravitation. Hence, the gravitation serves to “soften” the radiation force outward.

$$f_{net} = f_{rad} - f_G = \frac{\kappa L}{4\pi c R^2} - \frac{GM}{R^2}.$$

Let  $r$  = the non-initial position ( $R$  at some time  $t$ ). Then the total, potential, and kinetic energies are given by the expressions

$$E_{tot} = \frac{\kappa L}{4\pi c R} - \frac{GM}{R} \quad (5)$$

$$U = \frac{\kappa L}{4\pi c r} - \frac{GM}{r} \quad (6)$$

$$T = \frac{1}{2}v^2. \quad (7)$$

At terminal velocity,  $f_{net} = 0$ , so

$$\frac{\kappa L}{4\pi c r^2} - \frac{GM}{r^2} = 0 \Rightarrow GM = \frac{\kappa L}{4\pi c},$$

therefore (softened)  $U = 0$  for terminal velocity  $v_T$ .

Since  $E_{tot} = T - U$ ,

$$T = E_{tot} - U \quad (8)$$

$$\frac{1}{2}v_T^2 = \left( \frac{\kappa L}{4\pi c R} - \frac{GM}{R} \right) - 0$$

$$\frac{1}{2}v_T^2 = \frac{\kappa L}{4\pi c R} - \frac{GM}{R}$$

$$v_T^2 = \frac{2GM}{R} \left( \frac{\kappa L}{4\pi c GM} - 1 \right). \quad (9)$$

(c) The minimum value for  $\kappa$  may be estimated for pure hydrogen as that due to Thomson scattering off free electrons, when the hydrogen is completely ionized. The Thomson cross section is  $\sigma_T = 6.65 \times 10^{-25} \text{cm}^2$ . The mass scattering coefficient is therefore  $> \sigma_T/m_H$ , where  $m_H$  = mass of hydrogen atom. Show that the maximum luminosity

that a central mass  $M$  can have and still not spontaneously eject hydrogen by radiation pressure is

$$\begin{aligned} L_{EDD} &= 4\pi GMcm_H/\sigma_T \\ &= 1.25 \times 10^{38} \text{ ergs}^{-1} (M/M_\odot), \end{aligned}$$

where

$$M_\odot \equiv \text{mass of sun} = 2 \times 10^{33} \text{ g}.$$

This is called the Eddington limit.

Regardless of whether the radiation is being scattered or absorbed, the limiting case for ejection of material occurs when

$$\frac{M}{L} = \frac{\sigma_T/m_H}{4\pi cG}, \quad (10)$$

as can be seen by setting the term in parentheses in (b) equal to zero. In this limit,  $L \rightarrow L_{EDD}$ . Thus,

$$L_{EDD} = \frac{4\pi cGMm_H}{\sigma_T}. \quad (11)$$

Plugging in numbers for everything,

$$\begin{aligned} L_{EDD} &= \frac{4\pi(3 \times 10^{10} \text{ cms}^{-1})(6.673 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2})(1.674 \times 10^{-24} \text{ g})M}{6.65 \times 10^{-25} \text{ cm}^2} \\ L_{EDD} &= 6.33 \times 10^4 M \text{ s}^{-3} \text{ cm}^2, \end{aligned}$$

or, in terms of solar masses,

$$L_{EDD} = 1.27 \times 10^{38} \text{ ergs}^{-1} \cdot M/M_\odot.$$

Most objects in the universe, even the high-energy ones like accreting X-ray binaries and AGN, are radiating at sub-Eddington rates. I believe there are a few cases where compact objects are puzzlingly radiating above the Eddington limit; it has been conjectured that such objects are accreting through a thin disk, and radiating above and below the disk. That is, the inflow of matter and the outflow of radiation occur at different locations, so there is less danger of the radiation preventing the inflow.

## Problem 2. Pulsar Dispersion Measure

Don Backer notes that pulses from a certain pulsar observed at a radio frequency of  $\nu = 2 \text{ GHz}$  arrive slightly ahead of the pulse train observed at  $\nu = 1 \text{ GHz}$ . The lead time is 1 s.

(a) Use the dispersion relation derived in class for a cold, ionized plasma, and the information above, to derive the column density of electrons along the line of sight to this pulsar. Express in standard pulsar-community units of  $\text{cm}^{-3} \text{ pc}$ .

The dispersion relation (as derived in class) is given by

$$\left(\frac{ck}{\omega}\right)^2 = 1 + \frac{4\pi ne^2}{m_e(\omega_o^2 - \omega^2)}.$$

For a plasma,  $\omega_o^2 = 0$ , so the dispersion relation becomes

$$\left(\frac{ck}{\omega}\right)^2 = 1 - \frac{\omega_p^2}{\omega^2} \quad (12)$$

where

$$\omega_p^2 = \frac{4\pi ne^2}{m_e}.$$

Thus, the dispersion equation can be rewritten as

$$\omega = \sqrt{\omega_p^2 + k^2 c^2}. \quad (13)$$

Since the group velocity  $v_{group} = \frac{d\omega}{dk}$ ,

$$\begin{aligned} v_{group} &= \frac{d}{dk} \sqrt{\omega_p^2 + k^2 c^2} \\ v_{group} &= \frac{c^2 k}{\omega}. \end{aligned} \quad (14)$$

Since the wave number  $k = c^{-1} \sqrt{\omega^2 - \omega_p^2}$ ,  $v_{group}$  can be rewritten as

$$v_{group} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad (15)$$

Enter the pulsar. The group velocity of the 2 GHz signal is such that it arrives 1 second before the 1 GHz signal. Henceforth, I shall use  $\omega_1$  for the 2-GHz wave and  $\omega_2$  for the 1-GHz wave.

If it can be assumed that  $\omega_p \ll \omega$ , then the expression  $\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)^{-1/2}$  can be approximated as  $1 + \frac{1}{2} \left(\frac{\omega_p}{\omega}\right)^2$ . If  $s$  is the distance between Don and the pulsar, the time between signals can be expressed as

$$\Delta t = t_2 - t_1 = \frac{s}{v_{group,2}} - \frac{s}{v_{group,1}},$$

which becomes

$$\begin{aligned} \Delta t &= \frac{s}{c} \left[ 1 + \frac{1}{2} \left( \frac{\omega_p}{\omega_2} \right)^2 \right] - \frac{s}{c} \left[ 1 + \frac{1}{2} \left( \frac{\omega_p}{\omega_1} \right)^2 \right] \\ \left( \frac{2c}{s} \right) \Delta t &= \left( \frac{\omega_p}{\omega_2} \right)^2 - \left( \frac{\omega_p}{\omega_1} \right)^2. \end{aligned} \quad (16)$$

Since  $\omega_2 = \frac{1}{2}\omega_1$ ,

$$\left( \frac{2c}{3s} \right) \Delta t = \left( \frac{\omega_p}{\omega_1} \right)^2. \quad (17)$$

Replacing the expression for  $\omega_p^2$ ,

$$\left( \frac{4\pi n e^2}{m_e} \right) s = \frac{8\pi^2}{3} c v_1^2 \Delta t. \quad (18)$$

Cancelling convenient things like factors of  $\pi$  and plugging in  $\Delta t = 1$  s (with standard values for  $m_e$ , etc.), one ends up with a column density of

$$n \cdot s = 3 \times 10^2 \text{ cm}^{-3} \text{ pc}. \quad (19)$$

(b) For an assumed density of electrons in the interstellar medium of  $0.03 \text{ cm}^{-3}$ , calculate the distance to this pulsar. Does this seem reasonable?

The distance to the pulsar if  $n = 0.03 \text{ cm}^{-3}$ , is

$$\begin{aligned} (0.03 \text{ cm}^{-3}) s &= 3 \times 10^2 \text{ cm}^{-3} \text{ pc} \\ s &= 1 \times 10^4 \text{ pc} = 10 \text{ kpc} \end{aligned}$$

which is reasonable because it resides inside our Galaxy. I don't believe our Earth-bound detectors are sufficiently sensitive to detect extragalactic pulsars.

(c) For such an assumed electron density, is Don safely observing above the plasma cut-off frequency?

The plasma cutoff frequency is given by

$$\omega_p^2 = \frac{4\pi n e^2}{m_e} = \frac{4\pi(0.03\text{cm}^{-3})(4.8 \times 10^{-10}\text{esu})^2}{9.11 \times 10^{-28}\text{g}}$$
$$\omega_p = 10^4\text{s}^{-1}.$$

And  $\omega_1$  is

$$\omega_1 = 2\pi\nu_1 = 4\pi\text{GHz} = 10^{10}\text{s}^{-1}.$$

Thus

$$\frac{\omega_p}{\omega} = 10^{-6},$$

plenty small enough for Don to have observed the pulsar in the first place.

(d) Calculate the optical depth to Thomson scattering along the line-of-sight to this pulsar.

The optical depth due to Thomson scattering is given by

$$\tau = \sigma_T(n \cdot s) \tag{20}$$
$$\tau_T = (6 \times 10^{-25}\text{cm}^2)(3 \times 10^2\text{cm}^{-3}\text{pc}).$$

Taking  $1\text{pc} = 3 \times 10^{18}\text{cm}$ ,

$$\tau_T = 0.0005.$$

Thomson scattering attenuates the signal very little!

### **Problem 3.** *Hyperfine ${}^3\text{He}^+$*

*Observations of the hyperfine transition in  ${}^3\text{He}^+$  are used to probe the  ${}^3\text{He}/\text{H}$  abundance in the galaxy. This abundance reflects the primordial yield from big bang nucleosynthesis and galactic chemical evolution.*

(a) Estimate, using the scaling relations presented in class and whatever facts you remember, the wavelength of the ground-state hyperfine transition in  ${}^3\text{He}^+$ . Compare to the true answer of 3.46 cm.

This is a magnetic dipole transition. Energies of magnetic dipole transitions scale as  $E \sim \mu_e B$ , where  $\mu_e$  is the magnetic moment of the electron and  $B$  is the magnetic field strength experienced by the electron.

Let's scale from the more well-known 21-cm transition in hydrogen. Now  $\mu_e$ , the moment intrinsic to the electron, does not change in going from H to He. But  $B$  does change. For a dipole nuclear field,  $B \sim \mu_{\text{nucleus}}/r^3$ , where  $\mu_{\text{nucleus}}$  is the magnetic moment of the nucleus, and  $r$  is the distance between the nucleus and the electron. Both these quantities change in going from H to He.

Now the magnetic moment of a single proton is  $e\hbar/(2m_p c)$ . We infer that the magnetic moment of a nucleus of charge  $Ze$  and mass  $Am_p$  is  $\mu_{\text{nucleus}} = Ze\hbar/(2Am_p c)$ . The Bohr model tells us that  $r \sim 1/Z$ . Therefore  $B \propto (Z/A) \times Z^3$ . For our  ${}^3\text{He}$  nucleus,  $Z = 2$  and  $A = 3$ . Therefore the wavelength of our transition is  $21 \text{ cm} \times A/Z^4 \sim 3.9 \text{ cm}$ , which is pretty close to the laboratory-measured answer of 3.46 cm.

(b) Estimate the Einstein  $A$  coefficient (transition probability) of this line. Compare to the true answer of  $1.95 \times 10^{-12} \text{ s}^{-1}$ .

Scale again from the 21-cm transition in H, for which we know  $A_{21} \approx 2.9 \times 10^{-15} \text{ s}^{-1}$ . We know that Einstein  $A$ 's scale as some moment squared and frequency cubed. Now the moment appropriate to this problem is the magnetic moment of the electron which does not change in going from H to He. The frequency certainly does change by a factor of  $(21/3.9)$ . Then the Einstein  $A$  for this transition is about  $2.9 \times 10^{-15} \times (21/3.9)^3 \text{ s}^{-1} \sim 5 \times 10^{-13} \text{ s}^{-1}$ , which is a factor of 4 too low compared to the true answer. Had we used the true wavelength of 3.46 rather than our above estimate of 3.9 cm, we would be off by a factor of 3.

Here's an alternative scaling from the Ly  $\alpha$  electric dipole transition in H that just happens to get the answer almost exactly correct:

$$A_{21} \Big|_{\Delta E_{hf}, {}^3\text{He}^+} = A_{21} \Big|_{Ly\alpha} \alpha^2 \cdot \left( \frac{1216 \text{ \AA}}{3.46 \text{ cm}} \right)^3$$

$$A_{21} \Big|_{\Delta E_{hf}, {}^3\text{He}^+} = 1.3 \times 10^{-12} \text{ s}^{-1}.$$

#### Problem 4. Shades of the Sun

(a) When the sun sits 5 degrees above the horizon and is about to set, it looks red. Give a quantitative explanation why. Neglect dust.

Consider the Sun overhead, where the distance it passes through the atmosphere is given by  $H$ . The distance it passes through the atmosphere at the horizon (assuming that the horizon is a flat infinite slab) is  $H/\sin\theta$ , where  $\theta$  is the angle above the horizon.

Sunlight experiences Rayleigh scattering as it tries to propagate through the atmosphere. The cross-section for Rayleigh scattering is

$$\sigma_{scattering} = \sigma_{Thomson} \frac{\omega^4}{\omega_0^4}.$$

Compare the two extremes of the visible spectrum (taking  $\lambda_{red} = 7000\text{\AA}$  and  $\lambda_{blue} = 4000\text{\AA}$ ):

$$\frac{\sigma_{scat}(red)}{\sigma_{scat}(blue)} = \frac{\sigma_T \cdot \lambda_o^4 / \lambda_r^4}{\sigma_T \cdot \lambda_o^4 / \lambda_b^4} = \left(\frac{\lambda_b}{\lambda_r}\right)^4 = 0.11.$$

What is the absolute optical depth, say at blue wavelengths? At zenith, the column of (mostly nitrogen) molecules is  $N \sim 5 \times 10^{19} \text{ cm}^{-3} \times 10 \text{ km} \sim 5 \times 10^{25} \text{ cm}^{-2}$ . The cross-section for Rayleigh scattering at blue wavelengths is  $\sigma_{Thomson} \omega^4 / \omega_0^4$ , where  $\omega_0 \approx 10 \text{ eV} / \hbar$ . The 10 eV is the typical energy for an electronic transition in molecular nitrogen. Putting it all together, we get  $\tau_b \sim 0.2$ . Correct this by  $1/\sin 5^\circ$  to get  $\tau_b \sim 2.3$  towards the sun at sunset. Thus,  $e^{-2.3} \sim 10\%$  of the blue light reaches us.

Some of you wished to use the scattering law proved in the last problem set, but the geometry of this problem is different; for this problem, light that is scattered out of the incident beam remains out of the beam (3-D geometry), whereas in the previous problem set on clouds, light has no choice but to be scattered parallel or anti-parallel to the incident beam (1-D geometry). Our assumption here that light that is scattered out of the direction of a particular beam remains out of the beam is a good one because otherwise the disc of the sun would appear larger than just the geometric value, and we know this is not the case.<sup>1</sup> In other words, multiple scattering is not important for this problem; the extinction in this problem is still exponential; energy losses suffered by a given beam due to scattering are not regained by scattering from other directions.

By the ratio of cross-sections,  $\tau_r \sim 0.02$  towards the sun at sunset. Thus, about 98% of the red light reaches us.

And that's why sunsets are red: red and blue light are in the rough ratio of 10:1. Actually they'd be more orange, if weren't for all the particulates spewed by humanity that reddens the light even further.

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<sup>1</sup>That the sun and moon appear larger near the horizon than at zenith is, I believe, an optical illusion caused by the proximity of these celestial objects to terrestrial ones near the horizon.



(b) During a total solar eclipse, the sun's corona appears white to the naked eye. The corona is composed of plasma. State why the corona looks white (one sentence will suffice).

In a plasma,  $\omega_o = 0$ . So  $\sigma_{scattering} = \sigma_{Thomson}$ . Since  $\sigma_{scat}$  no longer has any wavelength dependence, all the light from the sun's photosphere is scattered evenly by the corona into our line-of-sight. Hence it appears "white."<sup>2</sup>

**Problem 5.** *Back-of-the-envelope CO 1-0*

Most astronomers know that the Einstein A coefficient for the Lyman alpha ( $n = 2$  to  $n = 1$ ) transition in atomic hydrogen is of order  $10^9 \text{ s}^{-1}$  (actually,  $5 \times 10^8 \text{ s}^{-1}$ , but what's a factor of 2 between friends?). We derived this result in class to order-of-magnitude by considering an (accelerating) electron on a spring that displaces a Bohr radius,  $a_0$ , and has natural angular frequency  $\omega$ . We took the inverse lifetime of the excited state as  $A \sim P/\hbar\omega$ , where  $P$  is the power radiated by an accelerating charge.

The ubiquitous carbon monoxide molecule,  $^{12}\text{CO}$ , is used by astronomers to trace the presence and temperature of molecular gas in everything from galaxies to circumstellar disks. We would rather try to detect  $\text{H}_2$ , but sadly  $\text{H}_2$  has no permanent dipole moment because of its symmetry. Carbon monoxide does have a permanent dipole moment.

(a) Estimate the wavelength of the lowest energy, rotational transition in CO ( $J = 1$  to  $J = 0$ , where  $J$  is the rotational quantum number). Do this by considering a barbell spinning about its axis of greatest moment of inertia and recognizing that angular momentum comes quantized in units of  $\hbar$ .

Compare your estimate to the true answer of 2.6 mm.

We know from quantum mechanics that angular momentum is quantized as  $L^2 = \hbar^2 J(J + 1)$ , and from classical mechanics we know the rotational kinetic energy of a barbell to be  $E = L^2/2I$ . Here, the moment of inertia  $I$  can be approximated as  $I = \sum_i m_i r_i^2 \sim 30m_p a_0^2$ , where  $m_p$  is the mass of a proton and we assume that the width of the barbell is roughly two times the Bohr radius. For transition  $J = 1$  to  $J = 0$  we find:

$$\Delta E_{1 \rightarrow 0} \sim \frac{\hbar^2}{30m_p a_0^2} \sim \frac{hc}{\lambda_{\text{CO}}}$$

which upon plugging in the appropriate values of the constants gives:

$$\lambda_{\text{CO}} \sim 2.6\text{mm}$$

which just happens to be exactly the same as the real value. Hooray for us!

(b) Use the scalings of our semi-classical spring model to estimate the Einstein A coefficient of this transition, i.e., the inverse lifetime of the excited  $J = 1$  state.

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<sup>2</sup>Actually, by this reasoning, it should appear slightly yellow, since yellow is the color of the sun's photosphere. Whether the corona is yellow or white I have yet to see with my own eyes.

Bring to bear one fact that is difficult to guess from first principles: the dipole moment of the CO molecule is 0.1 Debyes (1 Debye =  $10^{-18}$  cgs. Note that  $ea_0 = 2.5$  Debyes, where  $e$  is the electron charge), not  $\sim 1$  Debye, as one might have guessed naively. The smaller-than-usual dipole moment of CO is a consequence of the strong double bond connecting C to O. Most other molecules—e.g.,  $H_2O$ , CS, SiS, SiO, HCN, OCS,  $HC_3N$ —have dipole moments that are all of order 1 Debye.

Compare your estimate to the true answer of  $6 \times 10^{-8} s^{-1}$ .

We can estimate the Einstein A coefficient using the approximate equation derived in class:

$$A_{21} \sim \frac{2d^2\omega_0^3}{3\hbar c^3}.$$

Note we can just plug in our value of  $\lambda_{CO} = 2\pi c/\omega_0$  and standard constants for everything except for the dipole moment, which has a difficult to guess value of 0.1 Debye  $\sim 10^{-19}$ [cgs]. Plugging in these values gives:

$$A_{21} \sim 8.7 \times 10^{-8} s^{-1}$$

which is pretty close to the actual value of  $6 \times 10^{-8} s^{-1}$ .

(c) Estimate  $\sigma$ , the cross-section of the transition at line center, in  $cm^2$ . Assume the line to be thermally broadened at a typical molecular cloud temperature of 20 K.

To estimate the cross-section at line center of this transition, we apply the equation (again from lectures):

$$\sigma_{lc} \sim \frac{\lambda^2 A_{21}}{8\pi \Delta\nu}.$$

Here  $\Delta\nu$  is the broadening of the line. For our particular case, the broadening mechanism is thermal broadening, which gives  $\Delta\nu \sim \nu \frac{v_{\text{thermal}}}{c}$ . The thermal velocity is given by  $v_{\text{thermal}} = \sqrt{\frac{3kT}{m}}$ . Using  $T = 20K$  and  $m \sim 30m_p$  we find that the frequency broadening is  $\Delta\nu \sim 5 \times 10^4 s^{-1}$ . Plugging this into the cross-section equation gives

$$\sigma_{lc} \sim 5 \times 10^{-15} cm^2$$

(d) If the number abundance of CO molecules to  $H_2$  molecules is  $n_{CO}/n_{H_2} \sim 10^{-4}$  (i.e., close to the solar abundance ratio of  $n_C/n_H \sim 10^{-4}$ ), estimate the column of hydrogen molecules required to produce optical depth unity at the center of the CO line. Is this column likely to be exceeded in a molecular cloud? Take a typical molecular cloud  $H_2$  density of  $10^4 cm^{-3}$  and a cloud dimension of 100 pc.<sup>3</sup>

Assuming  $n_{H_2} \sim 10^4 cm^{-3}$  in molecular clouds (where we expect to see CO), and zero everywhere else, we use  $n_{CO}/n_{H_2} \sim 10^{-4}$  to find a rough number density of CO

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<sup>3</sup>In Spitzer's book on the interstellar medium, molecular cloud densities of molecular hydrogen are said to range from  $10^3$  to  $10^6 cm^{-3}$ . The density can be orders of magnitude higher than even  $10^6 cm^{-3}$  as you approach star forming regions. Moreover, cloud dimensions can range from 1 pc to 100 pc. Molecular clouds are clumpy structures.

in molecular clouds of 1 CO molecule per cubic centimeter. Using this number density and the cross-section from part (c), we expect to see an optical depth of unity when the column of CO molecules has a height  $s$  given by:

$$\tau = 1 = n_{\text{CO}}\sigma_{\text{lc}}s \text{ which gives } s \sim 2 \times 10^{14} \text{ cm} \sim 7 \times 10^{-5} \text{ pc.}$$

Compared with the rough size of a molecular cloud of 100 pc, we expect this column to be exceeded in our galaxy whenever the line of sight points at a molecular cloud.

Moral: Molecular clouds in the lowest-order ( $J = 1$  to  $J = 0$ ) rotational transition of the most common isotopomer of carbon monoxide,  $^{12}\text{CO}$ , are likely to be extremely optically thick. How do astronomers get around this problem to measure the mass of a molecular cloud? Some try using less common isotopomers of CO, such as  $^{13}\text{CO}$  (the terrestrial abundance of  $^{13}\text{C}$  to  $^{12}\text{C}$  is about 1/89) or  $\text{C}^{18}\text{O}$  (the terrestrial isotopic ratio is  $^{18}\text{O}/^{16}\text{O} \sim 1/500$ ). This helps, but not for the most massive of clouds that are considered here.

Others try looking at higher-order (higher  $J$ ) transitions in CO for which level populations will be smaller because molecular clouds are cold. And others give up on CO altogether and try to detect emission from optically thin dust (and then multiply by a factor of 100 to account for the gas-to-dust ratio) or other other, even less abundant molecules (whose abundance relative to hydrogen must be inferred from chemical modelling).

*(e) Based on your answer to (d), would you conclude that this transition is a good way to measure the mass of molecular gas in a galaxy? How about the temperature?*

If the line were merely thermally broadened, we could simply measure the width of the line and thereby infer the temperature. Real-world difficulties crop up, however, from non-thermal, bulk streaming motions (turbulence) in molecular clouds that complicate interpretation of a measured line width.

Because the clouds tend to be so optically thick in this transition, we are only sensitive to their outer skins, so an emission line measurement does not provide a good way to measure the mass of the most massive of clouds. In other words, we can't tell how much material lies beneath the surface of optical depth unity. See part (d) for alternative ways of measuring the masses of molecular clouds.