Astro 201 – Radiative Processes – Problem Set 4

Due in class.

Readings: However much of Rybicki & Lightman pages 77–90 that you wish; the goal is to feel comfortable with the Larmor power formula, Figure 3.2, near and far zones, and multipole expansions; you are free, of course, to achieve this goal by reading any text you like.

Problem 1. Hyperfine Emission from Neutral Hydrogen

This problem is an exercise in learning more astronomy jargon and in practicing some of the formalism in Rybicki & Lightman Chapter 1.

Neutral hydrogen in the electronic ground state can be in one of two hyperfine states. Denote the number density of atoms in the ground hyperfine level (singlet state) as n_0, and the number density of atoms in the excited hyperfine level (triplet state) as n_1. DEFINE the *excitation temperature*, T_{ex}, of the transition as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT_{ex}} \, . \tag{1}$$

Here $h\nu = hc/\lambda$ is the mean energy difference between the levels, and $g_0 = 1$ and $g_1 = 3$ are the statistical weights of the levels. The excitation temperature is merely another way of expressing the ratio of ground state to excited state populations. By definition, if a gas is in local thermodynamic equilibrium (LTE) at some gas kinetic temperature T, then $T_{ex} = T$; the level populations are distributed in Boltzmann fashion at the local temperature T. Some people refer to the excitation temperature for the $\lambda = 21\,\mathrm{cm}$ transition as the *spin temperature*. But use of the term “excitation temperature” is general to any line transition; it is simply a measure of how excited an atom is.

(a) Define $T_* = h\nu/k$ and compute its value.

It is likely that $T_{ex} \gg T_*$. For the remainder of this problem, work in the $T_*/T_{ex} \ll 1$ limit.

(b) Write down the absorption coefficient, α_ν (units of per length), for this transition. Express your answer in terms of $\phi(\nu)$ (the line profile function), A_{21} (Einstein A coefficient), λ, whatever densities you need, and T_*/T_{ex}. *Do not forget the correction for stimulated emission.*

(c) Write down the volume emissivity, j_ν (units of $\mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}\,sr^{-1}}$), for this transition. Use whatever quantities defined above that you need.

(d) Write down the source function, S_ν, for this transition.

(e) Write down the specific intensity, I_ν, of a cloud of HI that is optically thin along the line-of-sight (l-o-s). Take the l-o-s dimension of the cloud to be L, and give the answer only to leading order in $\tau \ll 1$, where τ is the optical depth at an arbitrary wavelength.

Does your answer depend on T_{ex}?

If someone gives you a spectrum of the 21 cm line that appears in emission and tells you that the line was emitted from an optically thin cloud, what physical quantities can you infer from the spectrum?

(f) Write down the optical depth of the cloud. Does your answer depend on T_{ex}?

(g) How large would L have to be for the cloud to be marginally optically thick? Use a gas density of $n = 1\,\mathrm{cm}^{-3}$, a gas temperature of $T = 100\,\mathrm{K}$, and an excitation temperature $T_{ex} = T$. Assume the line is only thermally broadened.

Problem 2. Multiple Multipoles

(a) Write down the electric field a distance r away from a monopole of charge q.

(b) Someone moves another monopole of charge $-q$ next to the original monopole. The two charges are separated by a distance b. Derive, to order-of-magnitude, the factor by which this dipole electric field is reduced from the monopole field.

(c) Someone moves two more charges, $-q$ and q, into position to form a square. The edge of the square has length b. Going around the square, the charges are $-q$, q, $-q$, and q. Derive, to order-of-magnitude, the factor by which this quadrupole electric field is reduced from the dipole field.

(d) Draw a picture of a pure electric octopole, that is, a charge distribution for which the electric field decreases as $1/r^5$ (and no less gradually). For bonus points, draw a picture of an electric hexadecapole (electric field dies no less gradually than $1/r^6$).

(e) Now imagine the dipole and quadrupole configurations in (b) and (c) rotating about their centers-of-charge with frequency ν. We have a rotating barbell and a rotating square, respectively. (Think CO and H_2).

To order-of-magnitude, what is the maximum distance from each object inside of which the electric fields are nearly perfectly in phase with the rotation?

This is the boundary of the *near zone*, inside of which the electric field geometry rotates with frequency ν as if it were a rigid body.

(f) Electromagnetic waves are emitted at the boundary of the near zone, into the far (radiation) zone.

Write down (one line of argument suffices) the factor by which the power carried by waves emitted by the rotating quadrupole is smaller than the power carried by waves emitted by the rotating dipole.

Problem 3. Rotating Magnetic Dipole

Rybicki & Lightman Problem 3.1.

Hint: The Larmor power formula gives the power radiated by an accelerating electric dipole moment. This problem is similar except that it concerns an accelerating magnetic dipole moment.