Ay 201 – Radiative Processes – Problem Set 4 Solutions

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Problem 1. Hyperfine Emission from Neutral Hydrogen

This problem is an exercise in learning more astronomy jargon and in practicing some of the formalism in Rybicki & Lightman Chapter 1.

Neutral hydrogen in the electronic ground state can be in one of two hyperfine states. Denote the number density of atoms in the ground hyperfine level (singlet state) as n_0, and the number density of atoms in the excited hyperfine level (triplet state) as n_1. DEFINE the excitation temperature, T_{ex}, of the transition as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT_{ex}} . \tag{1}$$

Here $h\nu = hc/\lambda$ is the mean energy difference between the levels, and $g_0 = 1$ and $g_1 = 3$ are the statistical weights of the levels. The excitation temperature is merely another way of expressing the ratio of ground state to excited state populations. By definition, if a gas is in local thermodynamic equilibrium (LTE) at some gas kinetic temperature T, then $T_{ex} = T$; the level populations are distributed in Boltzmann fashion at the local temperature T. Some people refer to the excitation temperature for the $\lambda = 21\,\text{cm}$ transition as the spin temperature. But use of the term "excitation temperature" is general to any line transition; it is simply a measure of how excited an atom is.

(a) Define $T_* = h\nu/k$ and compute its value.

It is likely that $T_{ex} \gg T_*$. For the remainder of this problem, work in the $T_*/T_{ex} \ll 1$ limit.

For the hyperfine transition in hydrogen ($\lambda = 21$cm),

$$T_* \equiv \frac{h\nu}{k} = \frac{hc}{\lambda k} = 0.067\,\text{K} \tag{2}$$

The excitation temperature is defined by the following:

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-\frac{h\nu}{kT_{ex}}} \tag{3}$$

with $g_0 = 1$ and $g_1 = 3$. We assume that $T_{ex} \gg T_*$. The next few parts take equations from Rybicki & Lightman pages 30 and 31.

(b) Write down the absorption coefficient, α_ν (units of per length), for this transition. Express your answer in terms of $\phi(\nu)$ (the line profile function), A_{21} (Einstein A coefficient), λ, whatever densities you need, and T_*/T_{ex}. Do not forget the correction for stimulated emission.

Including the correction for stimulated emission,

$$\alpha_\nu = \frac{h\nu}{4\pi}\phi(\nu)(n_0 B_{01} - n_1 B_{10}) = \frac{h\nu}{4\pi}n_0 B_{01}(1 - \frac{g_0 n_1}{g_1 n_0})\phi(\nu). \tag{4}$$

With $A_{10} = \frac{2h\nu^3}{c^2}\frac{g_0}{g_1}B_{01}$, equation 4 becomes

$$\frac{h\nu}{4\pi}n_0\frac{g_1}{g_0}\frac{c^2}{2h\nu^3}A_{10}(1 - e^{-\frac{h\nu}{kT_{ex}}})\phi(\nu) \tag{5}$$

$$= \frac{1}{4\pi}n_0 3\frac{c^2}{2\nu^2}A_{10}(1 - e^{-\frac{T_*}{T_{ex}}})\phi(\nu) \tag{6}$$

$$\approx \frac{3}{8\pi}n_0\lambda^2 A_{10}(1 - (1 - \frac{T_*}{T_{ex}}))\phi(\nu) \tag{7}$$

so

$$\alpha_\nu \approx \frac{3}{8\pi}n_0\lambda^2 A_{10}(\frac{T_*}{T_{ex}})\phi(\nu) \tag{8}$$

(c) Write down the volume emissivity, j_ν (units of $\mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}\,sr^{-1}}$), for this transition. Use whatever quantities defined above that you need.

$$j_\nu = \frac{h\nu_0}{4\pi}n_1 A_{10}\phi(\nu) \tag{9}$$

with $\nu_0 = \frac{c}{21\mathrm{cm}}$.

(d) Write down the source function, S_ν, for this transition.

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} = \frac{h\nu_0}{4\pi}n_1 A_{10}\phi(\nu)\left(\frac{3}{8\pi}n_0\lambda^2 A_{10}(\frac{T_*}{T_{ex}})\phi(\nu)\right)^{-1} \tag{10}$$

$$= \frac{2}{3}h\nu_0\frac{n_1}{n_0}\lambda^{-2}\left(\frac{T_{ex}}{T_*}\right) = \frac{2}{3}h\nu_0 3e^{-\frac{h\nu}{kT_{ex}}}\lambda^{-2}\left(\frac{T_{ex}}{T_*}\right) \approx 2h\nu_0\lambda^{-2}\frac{T_{ex}}{T_*} \tag{11}$$

(e) Write down the specific intensity, I_ν, of a cloud of HI that is optically thin along the line-of-sight (l-o-s). Take the l-o-s dimension of the cloud to be L, and give the answer only to leading order in $\tau \ll 1$, where τ is the optical depth at an arbitrary wavelength.

Does your answer depend on T_{ex}?

If someone gives you a spectrum of the 21 cm line that appears in emission and tells you that the line was emitted from an optically thin cloud, what physical quantities can you infer from the spectrum?

The source function S_ν is constant throughout the uniform HI cloud, so S_ν is independent of τ_ν.

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \tag{12}$$

Since we want to know the specific intensity of the cloud itself, we can say that $I_\nu(0) = 0$ (there is no light entering the cloud). We can also apply our assumption of an optically thin cloud ($\tau_\nu \ll 1$ for all ν).

$$I_\nu(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu}) \approx S_\nu(1 - (1 - \tau_\nu)) = S_\nu \tau_\nu \tag{13}$$

Now we use the definition of τ_ν and the fact that we want the specific intensity of light emerging from the cloud, meaning an optical depth corresponding to a path length L.

$$I_\nu(L) \approx S_\nu n_\nu \sigma_\nu L = S_\nu \alpha_\nu L = j_\nu L \tag{14}$$

$$= \frac{h\nu_0}{4\pi} n_1 A_{10} \phi(\nu) L \tag{15}$$

The specific intensity thus has no dependence on T_{ex}.

Now ν_0 and A_{10} are known quantities from the lab/quantum mechanics. A spectrum would tell us I_ν and $\phi(\nu)$, so we could find $n_1 L$, the column density of excited hydrogen atoms. Assuming $T_{ex} \gg T_*$, $n_0 \approx \frac{g_0}{g_1} n_1 = \frac{1}{3} n_1$, so we can find $n_0 L$ as well (the column density of ground state hydrogen atoms). We can add these column densities to get the total column density of hydrogen atoms:

$$nL = (n_1 + n_0)L = \frac{4}{3} n_1 L = \frac{16\pi}{3} (h\nu_0)^{-1} A_{10}^{-1} (\phi(\nu))^{-1} I_\nu \tag{16}$$

Knowing the column density gives us the mass if the cloud is roughly spherical and we know its length.

Finally, some idea of the temperature and/or velocity field of the gas could be gained by looking at the line shape (line profile).

(f) Write down the optical depth of the cloud. Does your answer depend on T_{ex}?

Well,

$$\tau_\nu = n_\nu \sigma_\nu L = \alpha_\nu L = \frac{3}{8\pi} n_0 \lambda^2 A_{10} \left(\frac{T_*}{T_{ex}}\right) \phi(\nu) L \tag{17}$$

The optical depth does depend on T_{ex}.

(g) How large would L have to be for the cloud to be marginally optically thick? Use our canonical gas density of $n = 1\,\mathrm{cm}^{-3}$, a gas temperature of $T = 100\,\mathrm{K}$, and an excitation temperature $T_{ex} = T$. Assume the line is only thermally broadened.

The condition for marginal optical thickness is $\tau \sim 1$.

$$n = n_1 + n_0 \approx 4n_0 \Rightarrow n_0 \approx \frac{1}{4} n \tag{18}$$

$$1 = \frac{3}{8\pi} n_0 \lambda^2 A_{10} \left(\frac{T_*}{T_{ex}}\right) \phi(\nu) L \tag{19}$$

$$\Rightarrow L \sim \frac{32\pi}{3n} \lambda^{-2} A_{10}^{-1} \left(\frac{T_{ex}}{T_*}\right) \phi(\nu)^{-1} \tag{20}$$

To find L, we evaluate equation 20 at $\lambda = 21\mathrm{cm}$. The Einstein A coefficient $\sim 3 \times 10^{-15} s^{-1}$ at 21 cm for this transition, we use T_* from part (a), and take $\phi(\nu) \sim 1/\Delta\nu \sim c/(v\nu_0) \sim \lambda/v$ at line center, where $v \sim 1\,\mathrm{km/s}$ for $T = 100\,\mathrm{K}$ gas. Then

$$L \sim 50\,\mathrm{pc} \tag{21}$$

Problem 2. Multiple Multipoles

(a) Write down the electric field a distance r away from a monopole of charge q.

q/r^2

(b) Someone moves another monopole of charge $-q$ next to the original monopole. The two charges are separated by a distance b. Derive, to order-of-magnitude, the factor by which this dipole electric field is reduced from the monopole field.

A first-order Taylor expansion gives the reduction factor b/r.

(c) Someone moves two more charges, $-q$ and q, into position to form a square. The edge of the square has length b. Going around the square, the charges are $-q$, q, $-q$, and q. Derive, to order-of-magnitude, the factor by which this quadrupole electric field is reduced from the dipole field.

A second-order Taylor expansion gives the reduction factor b/r.

(d) Draw a picture of a pure electric octopole, that is, a charge distribution for which the electric field decreases as $1/r^5$ (and no less gradually). For bonus points, draw a picture of an electric hexadecapole (electric field dies no less gradually than $1/r^6$).

You can draw the octopole by placing charges on the vertices of a cube, with alternating signs.

Here's another way to draw an octopole: draw a circle, and arrange 8 charges symmetrically on the circle with alternating signs. I spent part of Thanksgiving holiday verifying that this last configuration indeed gives a field that dies as b^3/r^5 along an axis that overlays a diameter of this circle intersecting two of the charges. So one can construct an octopole without having to go into the 3rd dimension!

Presumably, one can construct an electric hexadecapole by drawing a circle, and arranging 16 charges symmetrically on the circle with alternating signs. (But I haven't checked explicitly that such a configuration gives a field that dies as b^4/r^6.)

(e) Now imagine the dipole and quadrupole configurations in (b) and (c) rotating about their centers-of-charge with frequency ν. We have a rotating barbell and a rotating square, respectively. (Think CO and H_2).

To order-of-magnitude, what is the maximum distance from each object inside of which the electric fields are nearly perfectly in phase with the rotation?

This is the boundary of the near zone, inside of which the electric field geometry rotates with frequency ν as if it were a rigid body.

The information that the charge distribution has rotated propagates at the speed of light away from the distribution. Therefore a distance $\sim c/\nu$ marks the boundary of the near zone.

(f) Electromagnetic waves are emitted at the boundary of the near zone, into the far (radiation) zone.

Derive (one line of argument suffices) the factor by which the power carried by waves emitted by the rotating quadrupole is smaller than the power carried by waves emitted by the rotating dipole.

The waves are launched at the boundary of the near zone.

Since the power carried by electromagnetic waves goes as the square of the field, we must square the reduction factor for the field. In other words, the power reduction factor is $(b/r)^2$, where $r \sim c/\nu \sim \lambda$ is the size of the near zone. Then the power reduction factor is $(b/\lambda)^2$, as used throughout this class.

Problem 3. Rotating Magnetic Dipole

Rybicki & Lightman Problem 3.1

See solution 3.1 at back of the book.