Astronomy 201 – Radiative Processes – Solution Set 5

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Readings: Hand-outs from Osterbrock; Rybicki & Lightman 9.5; however much you like of Mihalas 108-114, 119-127, 128-137 (even skimming Mihalas can prove enlightening); however much you like of the hand-out of Purcell & Field (1957).

Problem 1: Maxwellian is not quite LTE

A certain atom suffers collisions with another species (perhaps merely its own). These collisions populate and de-populate a certain level in the atom that lies above another level in that atom by energy E. Prove that if the relative velocity distribution, $f(v)$, between the atom and surrounding colliders obeys a Maxwellian at temperature T ($\int f(v)dv = 1$), then the excitation rate coefficient,

$$q_{12} = \int \sigma_{12}(v) f(v) v dv$$

is related to the de-excitation rate coefficient,

$$q_{21} = \int \sigma_{21}(v) f(v) v dv$$

by

$$q_{12} = q_{21} \frac{g_2}{g_1} e^{-E/kT}$$

where g_i is the statistical weight of level i, σ_{ij} is the velocity-dependent collisional cross-section for making a transition from level i to level j, and v is the relative velocity between the atom and the colliding species. You should find this problem easy given the Einstein analogue presented in lecture.

NOTE: Nowhere in this discussion have we assumed that the level populations are distributed in a Boltzmann fashion at temperature T. That is, none of the relations above assume LTE (local thermodynamic equilibrium); LTE is a more restrictive condition than merely assuming that the relative velocity distribution is Maxwellian.

$$\text{show: } \quad q_{12} = \frac{g_2}{g_1} e^{-E/kT} q_{21}$$

$$\text{font note: } \quad v = \textit{velocity}, \qquad \upsilon = \textit{frequency}$$

$$\text{start with the Einstein analogue: } \quad g_1 \sigma_{12}(v) v^2 = g_2 \sigma_{21}(v') v'^2$$

$$\text{Multiply both sides by: } 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mv^2}{2kT} \right)$$

$$g_1 \sigma_{12}(v) v^2 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mv^2}{2kT} \right) = g_2 \sigma_{21}(v') v'^2 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mv^2}{2kT} \right)$$

$$\text{remember: } \tfrac{1}{2} mv^2 = \tfrac{1}{2} mv'^2 + E$$

$$\text{so} \ldots \quad \frac{-1}{kT}\left(\tfrac{1}{2} mv^2 \right) = \frac{-1}{kT}\left(\tfrac{1}{2} mv'^2 + E \right) = \frac{-mv'^2}{2kT} - \frac{E}{kT}$$

$$g_1 \sigma_{12}(v) v^2 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mv^2}{2kT} \right) = g_2 \sigma_{21}(v') v'^2 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mv'^2}{2kT} \right) \exp\left(\frac{-E}{kT} \right)$$

$$\text{isn't it lucky that: } \quad f(v) = v^2 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(\frac{-mv^2}{2kT} \right) \quad \text{so} \ldots$$

$$g_1 \sigma_{12}(v) f(v) = g_2 \sigma_{21}(v') f(v') e^{-E/kT}$$

now multiply by *vdv* and integrate both sides

$$\int g_1 \sigma_{12}(v) f(v) v dv = \int g_2 \sigma_{21}(v') f(v') e^{-E/kT} v dv$$

$$\text{remember: } vdv = v'dv'$$

$$\int g_1 \sigma_{12}(v) f(v) v dv = \int g_2 \sigma_{21}(v') f(v') e^{-E/kT} v' dv'$$

$$\int \sigma_{12}(v) f(v) v dv = \frac{g_2}{g_1} e^{-E/kT} \int \sigma_{21}(v') f(v') v' dv$$

$$\text{remember: } \quad q_{12} = \int \sigma_{12}(v) f(v) v dv \quad \text{and} \quad q_{21} = \int \sigma_{21}(v) f(v) v dv$$

$$\text{so} \ldots \quad q_{12} = \frac{g_2}{g_1} e^{-E/kT} q_{21}$$

Problem 2: 21 (flavors of) Temperatures for 21cm radiation

This problem continues our examination of the famous hyperfine transition in neutral hydrogen, begun in problem set 4. Here we try to understand what sets the excitation (spin) temperature, T_{ex}. (In the last problem set, we merely contented ourselves with the statement that $T_{ex} >> T_*$) While we're at it, we learn more astronomer's jargon.

In general, the excitation temperature of a transition is influenced by two factors: (1) the radiation field, and (2) collisions with surrounding species.

The radiation field can either excite the atom through photon absorption, or de-excite through stimulated emission. Measure the strength of the ambient radiation field at 21 cm by $\bar{J}$, the mean (i.e., angle-averaged) intensity integrated over the hyperfine line profile (recall Rybicki & Lightman chapter 1).

As for collisions, consider here exciting and de-exciting collisions with fellow neutral hydrogen atoms; Purcell and Field (1957, hereafter PF) conclude that collisions between a given electronic-ground-state H atom and other electronic-ground-state H atoms are most important in the predominantly neutral HI clouds of the ISM. (Electrons are 42 times faster and tend to dominate the excitation dynamics, but assume there are too few of them in these cold clouds.)
Denote the collisional excitation rate coefficient by q_{12}, and the collision de-excitation rate coefficient by q_{21} (see problem 1). Assume for this problem that both hyperfine-excited and hyperfine-ground atoms can excite or de-excite the hyperfine level in an atom.

(a) Write down the equation of global (not detailed!) balance for this transition. That is, write down the statement that the rate of excitations (from all possible channels) per volume per time equals the rate of de-excitations (from all possible channels) per volume per time.

Use only the following variables: n_1 and n_2 are the number densities of atoms in the ground and excited states, respectively, $n = n_1 + n_2$, T_K is the kinetic temperature of the atoms that move according to a Maxwellian, q_{12}, any Einstein coefficients you want, $\bar{J}$, and the statistical weights g_1 and g_2 of the ground and excited states, respectively.

What you have written down is an equation for the excitation temperature ($T_{ex} \leftrightarrow n_1/n_2$) in terms of the radiation field and the rate of collisions. Regard the latter two as given throughout this problem.

$$R_{\text{absorption}} + R_{\text{collisional\_excitation}} = R_{\text{spontaneous\_emission}} + R_{\text{stimulated\_emission}} + R_{\text{collisional\_de-excitation}}$$

$$n_1 B_{12}\bar{J} + n_1 n q_{12} = n_2 A_{21} + n_2 B_{21}\bar{J} + n_2 n q_{21}$$

$$n_1\left(B_{12}\bar{J} + nq_{12}\right) = n_2\left(A_{21} + B_{21}\bar{J} + nq_{21}\right)$$

$$\frac{n_1}{n_2} = \frac{\left(A_{21} + B_{21}\bar{J} + nq_{21}\right)}{\left(B_{12}\bar{J} + nq_{12}\right)}$$

$$\text{from \#1}: \quad q_{21} = \frac{g_1}{g_2} e^{T_*/T_K} q_{12}$$

$$\frac{n_1}{n_2} = \frac{\left(A_{21} + B_{21}\bar{J} + nq_{12}\dfrac{g_1}{g_2}e^{T_*/T_K}\right)}{\left(B_{12}\bar{J} + nq_{12}\right)}$$

(b) DEFINE a "radiation temperature," T_R, from $\bar{J}$ as

$$\bar{J} = B_\upsilon(T_R)$$

Note that we are NOT saying the ambient radiation field is Planckian. We are merely DEFINING a number T_R by using Planck's function, B_υ, where υ = 1420MHz, the frequency of the 21 cm line.

Re-write your equation in (a) to solve for T_{ex} in terms of the following variables: T_K, T_R, $T_* \equiv h\nu/k$ (recall last problem set), and the dimensionless variable

$$z = \frac{g_1}{g_2}\frac{nq_{12}}{A_{21}}\frac{T_*}{T_K}$$

where $A_{21} = 2.85x10^{-15}\ s^{-1}$ is the Einstein decay coefficient. Use the very likely condition that $T_K, T_R >> T_*$ to rid your equation of all exponentials.

Verify that if $z >> 1$, $T_{ex} \approx T_K$ (collisions beat radiation; the transition is in LTE at T_K), but that if $z << 1$, $T_{ex} \approx T_R$ (radiation beats collisions; the transition is not in LTE at T_K).

$$\text{note}: \quad T_* << T_R \quad \textit{and} \quad T_* << T_K$$

$$B_{21} \equiv \frac{c^2}{2h\upsilon^3} A_{21} \qquad B_{12} \equiv \frac{g_2}{g_1}\frac{c^2}{2h\upsilon^3} A_{21} \qquad \bar{J} = \frac{2h\upsilon^3}{c^2\left[\exp(T_*/T_R)-1\right]} \approx \frac{2h\upsilon^3}{c^2}\frac{T_R}{T_*}$$

we now see why the form of $\bar{J}$ was chosen

$$\frac{n_1}{n_2} = \frac{A_{21} + A_{21}\frac{T_R}{T_*} + nq_{12}\frac{g_1}{g_2}\left(\frac{T_*}{T_K}+1\right)}{\left(A_{21}\frac{g_2}{g_1}\frac{T_R}{T_*} + nq_{12}\right)} = \frac{g_1}{g_2}\exp(T_*/T_{ex}) \approx \frac{g_1}{g_2}\left(1+\frac{T_*}{T_{ex}}\right)$$

$$1+\frac{T_*}{T_{ex}} = \frac{A_{21}\left[1+\frac{T_R}{T_*}+\frac{g_1}{g_2}\frac{nq_{12}}{A_{21}}\left(\frac{T_*}{T_K}+1\right)\right]}{A_{21}\frac{g_1}{g_2}\left(\frac{g_2}{g_1}\frac{T_R}{T_*}+\frac{nq_{12}}{A_{21}}\right)} = \frac{1+\frac{T_R}{T_*}+\frac{g_1}{g_2}\frac{nq_{12}}{A_{21}}\left(\frac{T_*}{T_K}+1\right)}{\left(\frac{T_R}{T_*}+\frac{g_1}{g_2}\frac{nq_{12}}{A_{21}}\right)}$$

$$\text{let } z = \frac{g_1}{g_2}\frac{nq_{12}}{A_{21}}\frac{T_*}{T_K} \quad \text{so}\ldots \quad 1+\frac{T_*}{T_{ex}} = \frac{1+\frac{T_R}{T_*}+z+z\left(\frac{T_K}{T_*}\right)}{\frac{1}{T_*}\left(T_R+zT_K\right)}$$

$$\frac{1}{T_{ex}} = \frac{T_*\left[1+\frac{T_R}{T_*}+z+z\left(\frac{T_K}{T_*}\right)\right]-\left(T_R+zT_K\right)}{T_*\left(T_R+zT_K\right)} = \frac{T_*+T_R+zT_*+zT_K-\left(T_R+zT_K\right)}{T_*\left(T_R+zT_K\right)}$$

$$\frac{1}{T_{ex}} = \frac{1+z}{T_R+zT_K} \quad \text{so}\ldots \quad T_{ex} = \frac{T_R+zT_K}{1+z}$$

If $z >> 1$, $T_{ex} \approx zT_K/z = T_K$

If $z << 1$, $T_{ex} \approx T_R$

(c) To order of magnitude (actually much better than that), what fraction of HI is in the excited hyperfine state? Recall that $g_1 = 1$ and $g_2 = 3$ and use the very likely condition that T_K, $T_R >> T_*$.

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp(-T_*/T_{ex}) \quad \text{if we assume :} \quad T_* << T_{ex}$$

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} = 3 \quad so\ldots \quad n_2 = 3n_1$$

$$\frac{n_2}{n} = \frac{n_2}{n_1 + n_2} = \frac{3n_1}{n_1 + 3n_1} = 0.75$$

(d) Estimate the value for z for an HI cloud at $T_K = 100\text{K}$, $n = 1\ \text{cm}^{-3}$. Use Table 1 and equation (9) of PF; note that PF's collision frequency $\nu = n\langle\sigma v\rangle$ is not the same as our line frequency ν; call PF's $\nu = \nu_{PF}$; then $nq_{12} = 3\ \nu_{PF} / 8$. (For those interested, the 3/8 can be understood easily; skim the first 3 pages of PF and use your answer for part (c).)

Based on your answer, would you expect collisions or radiation to be more important in determining the degree of excitation?

$$z = \frac{g_1}{g_2}\frac{nq_{12}}{A_{21}}\frac{T_*}{T_K} = \frac{g_1}{g_2}\frac{\frac{3}{8}\upsilon_{pf}}{A_{21}}\frac{h\upsilon}{kT_K} = \frac{3}{8}\frac{g_1}{g_2}\frac{\sqrt{2}n\bar{\sigma}\left(\frac{8kT_K}{\pi M}\right)^{1/2}}{A_{21}}\frac{h\upsilon}{kT_K}$$

$$= \frac{(1)\sqrt{2}\left(5.65\times10^{-15}\right)\left(6.6\times10^{-27}\right)\left(1.42\times10^{9}\right)}{8\left(2.85\times10^{-15}\right)\left(1.38\times10^{-16}\right)(100)}\left(\frac{8\left(1.38\times10^{-16}\right)100}{\pi\left(1.67\times10^{-24}\right)}\right)^{1/2}$$

$$= 34.43$$

$z>>1$, so the system is collisional

(e) "Critical densities," n_{crit}, for exciting the line by collisions are defined by setting the rate of radiative de-excitations equal to the rate of collisional de-excitations. Show that such a procedure gives

$$n_c = \frac{A_{21}}{q_{21}}$$

and solve for its value for this line at $T_K = 100\text{K}$.

One can define n_{crit} for any line transition at any temperature; it is a crude gauge of the density of colliders required for collisions to be important in exciting the line.

ignore stimulated emission

$$n_2 A_{21} = n_2 n_{crit} q_{21}$$

$$n_{crit} = \frac{A_{21}}{q_{21}}$$

$$n_{crit} = \frac{A_{21}}{\frac{g_1}{g_2} e^{T_*/T_K} q_{12}} = \frac{g_2}{g_1} \frac{A_{21}}{\frac{3}{8}n\, \upsilon_{pf} e^{T_*/T_K}} = \frac{g_2}{g_1} \frac{8}{3} \frac{A_{21}}{\sqrt{2}\bar{\sigma}\left(\frac{8kT_K}{\pi M}\right)^{1/2} e^{h\upsilon/kT_K}} =$$

$$= \frac{8\left(2.85\times10^{-15}\right)}{\sqrt{2}\left(5.65\times10^{-15}\right)\left(\frac{8\left(1.38\times10^{-16}\right)100}{\pi\left(1.67\times10^{-24}\right)}\right)^{1/2} \exp\left(\frac{\left(6.6\times10^{-27}\right)\left(1.42\times10^{9}\right)}{\left(1.38\times10^{-16}\right)100}\right)}$$

$$= 2\times10^{-5}\, cm^{-3}$$

(f) Suppose radio observations are made that spatially resolve emission from a uniform HI cloud that is optically *thick* to its own 21 cm line radiation. Prove that the observed specific intensity, I_ν equals

$$I_\upsilon = \frac{2kT_{ex}}{\lambda^2}$$

optically thick, so use S_υ

$$I_\upsilon \approx S_\upsilon = \frac{j_\upsilon}{\alpha_\upsilon} = \{see\ RL\ eq\ 1.79\} = \frac{2h\upsilon^3}{c^2}\left(\frac{g_2}{g_1}\frac{n_1}{n_2} - 1\right)^{-1} = \frac{2h\upsilon^3}{c^2}\left(\frac{g_2}{g_1}\frac{g_1}{g_2} e^{T_*/T_{ex}} - 1\right)^{-1}$$

$$\approx \frac{2h\upsilon^3}{c^2}\left(1 + \frac{T_*}{T_{ex}} - 1\right)^{-1} = \frac{2h\upsilon^3}{c^2}\frac{T_{ex}}{T_*} = \frac{2h\upsilon^3}{c^2}\frac{kT_{ex}}{h\upsilon} = \frac{2\upsilon^2 kT_{ex}}{c^2} = \frac{2kT_{ex}}{\lambda^2}$$

Problem 3: Photoionized Quasar winds

Quasars are luminous X-ray sources sitting in the cores of ancient galaxies. They are supermassive black holes that accrete surrounding gas; the gravitational potential energy of gas spiralling down the potential well of the black hole is converted into radiation. The luminosity of a typical quasar is $L \approx 10^{46}\ erg\ s^{-1}$, mostly in the Lyman continuum, with a substantial fraction in a power-law X-ray tail. The flux density in the X-ray tail obeys $F_\nu \propto \nu^{-\beta}$, with $1 < \beta < 2$. (F_ν has units of energy per time per frequency per area).

This radiation streams out from regions closest to the black hole and may illuminate more distant but still circumnuclear gas. In so doing, it photo-ionizes the more distant gas and threatens to turn it into a complete and utter plasma, with *all* electrons stripped from parent nuclei. This problem estimates the varying degrees to which this threat is made good. It is inspired by work done by Norm Murray.

(a) Define the "ionization parameter," ξ, as the number density of Lyman continuum photons, η, divided by the total number density of hydrogen, $n_H = n_{H^+} + n_{H^0}$, where n_{H^+} is the number density of ionized hydrogen, and n_{H^0} is the number density of neutral hydrogen. Show that in photo-ionization equilibrium (rate of photo-ionizations per volume per time equals the rate of radiative recombinations per volume per time):

$$\frac{N_0}{N_+} = \frac{10^{-6}}{\xi}$$

valid for $n_{H^0} / n_{H^+} << 1$. This is a rule of thumb worth remembering. Just consider photons near the ionization edge! Furthermore, assume a 1-dimensional geometry for the problem; consider only a semi-infinite slab, upon which is incident a radiation flux. In your derivation, you will see that the dimensionless factor of 10^{-6} can be expressed in terms of fundamental constants.

$$\text{note :} \quad \eta = \frac{u}{h\upsilon} \quad \xi = \frac{\eta}{N}$$

$$N_0 4\pi \int \frac{\sigma_{bf}}{h\upsilon} J_\upsilon d\upsilon \rightarrow 4N_0 \frac{\sigma_{bf}}{h\upsilon} F = \alpha N_+ N_e$$

$$4N_0\frac{\sigma_{bf}}{h\upsilon}F = 4N_0\frac{c}{4}\frac{\sigma_{bf}}{h\upsilon}F\frac{4}{c} = 4N_0\sigma_{bf}\frac{c}{4}\frac{u}{h\upsilon} = N_0\sigma_{bf}c\frac{u}{h\upsilon} = N_0\sigma_{bf}c\eta = \alpha N_+ N_e$$

$$\frac{N_0}{N_+} = \frac{\alpha N_e}{c\sigma_{bf}\eta}$$

now look up α and σ_{bf} in tables

$$\frac{N_0}{N_+} = \frac{6.8\times10^{-13}}{\left(3\times10^{10}\right)\left(6.3\times10^{-18}\right)}\frac{N_e}{\eta} \approx 4\times10^{-6}\frac{N}{\eta} \approx \frac{10^{-6}}{\xi}$$

(b) Prove for hydrogenic ions of species X and nuclear charge *Z* (atoms that are holding on desperately to their last electron) that

$$\frac{N_x^0}{N_x^+} \approx \frac{10^{-6}}{\xi}z^{2\beta+4}$$

where n_{X^0} is the number density of hydrogenic ions of species X that each still retain their last electron, and n_{X^+} is the number density of fully stripped ions of species X. Assume that all of the hydrogenic metal ions are recombining with electrons provided by hydrogen, and that nearly all of the hydrogen is ionized. We check this last statement in part (d).

Steps you can take:

Use the Bohr model to understand how ionization energy scales with *Z*.

Use our dirty derivation of the photo-ionization cross-section,

$$\sigma \approx \frac{\lambda^2}{8\pi}\frac{A_{21}}{\upsilon},$$

to argue that σ scales as the radius of the hydrogenic ion squared. Then use the Bohr model to see how this radius scales with *Z*.

Finally, once you have determined how σ scales with *Z*, use the Milne relation to see how the radiative recombination coefficient scales with *Z*.

(1) Bohr atomic radius : $a \propto 1/z$

(2) Bound Free crosssection:

$$\sigma_{bf} \propto A_{21}\lambda^3$$

$$A_{21} \propto \upsilon^3 d^2 \propto \frac{d^2}{\lambda^3}$$

$$d(\equiv \text{dipole moment}) \propto a$$

$$\sigma_{bf} \propto a^2 \propto \frac{1}{z^2}$$

(3) Ionization Energy: $h\upsilon = ze^2/a \rightarrow \upsilon \propto z^2$

(4) Milne: $\sigma_{fb} \propto \sigma_{bf}\upsilon^2 \propto \frac{1}{z^2}\left(z^2\right)^2 \propto z^2$

(5) $$N_x^0\left(\frac{\upsilon F_\upsilon}{h\upsilon}\right)\sigma_{bf} \approx N_x^+ N_e \sigma_{fb} f(v)v$$

$$\left(\frac{\upsilon F_\upsilon}{h\upsilon}\right) \equiv \text{Number Flux of ionizing photons} \propto F_\upsilon \propto \upsilon^{-\beta} \propto \left(z^2\right)^{-\beta} \propto z^{-2\beta}$$

$$\therefore N_x^0\left(\frac{\upsilon F_\upsilon}{h\upsilon}\right)\sigma_{bf} \propto N_x^0 z^{-2\beta}\eta z^{-2} \qquad \& \qquad N_x^+ N_e \sigma_{fb} \propto N_x^+ n_H z^2$$

$$N_x^0 z^{-2\beta}\eta z^{-2} \propto N_x^+ n_H z^2$$

All the constants are the same as in (a) so...

$$\frac{N_x^0}{N_x^+} \approx \frac{10^{-6}}{\xi} z^{2\beta+4}$$

(c) Notice that no matter how large ξ is, there are always a few electrons bound to nuclei at any given moment (if there were none, there would be nothing for photons to ionize, and photo-ionization equilibrium would be violated). This tiny neutral/hydrogenic population attenuates the UV-to-X-ray radiation as it tries to propagate through gas. The gas will have an ionization gradient: at its unshielded face, naked before the radiation, nearly all the ions will be stripped, but as we move further away from the face, the ionizing radiation weakens due to increasing absorption by the tiny neutral/hydrogenic population, and the neutral/hydrogenic fraction grows.

Show that an element of fractional number abundance $f_x = n_X / n_H$ relative to hydrogen ($n_X = n_{X^+} + n_{X^0}$) goes from being fully stripped to predominantly hydrogenic over a column density of *hydrogen* of order

$$N_H(Z) \approx 10^{23}\xi z^{-2\beta-2} f_x^{-1} cm^{-2}$$

Assume that such columns are achieved over regions sufficiently geometrically thin that we can neglect the dilution of the radiation flux by the inverse square law. Again, consider only ionizations near the ionization edges of various species.

$$note: \quad N_H(z) = n_H L \quad (number\ density * depth)$$

$$\tau = n_x^0 \sigma_{bf,x} L \approx 1 \quad \rightarrow \quad n_x^0 L = \frac{1}{\sigma_{bf,x}}$$

$$N_H(Z) = n_H L = n_H L \frac{n_x^0}{n_x^0} = n_x^0 L \frac{n_H}{n_x^0} \approx \frac{1}{\sigma_{bf,x}} \frac{n_H}{n_x^0} = \frac{z^2}{\sigma_{bf,H}} \frac{n_H}{n_x} \frac{n_x^+}{n_x^0} =$$

$$= \frac{z^2}{6.3\times10^{-18} cm^2} f_x^{-1} 10^6 \xi z^{-2\beta-4} \approx 10^{23} \xi z^{-2\beta-2} f_x^{-1} cm^{-2}$$

(d) If β > 1.5, show that the layers of fully stripped carbon, nitrogen, oxygen, and neon are each thinner than the layer of fully stripped hydrogen. In other words, soft X-rays are stopped by the metals before the Lyman continuum photons are stopped by hydrogen. This is a fact of relevance in understanding how gas can be radiatively accelerated to form quasar winds.

$$N_H(Z) \propto z^{-2\beta-2} f_x^{-1} \propto Thickness \equiv P$$

$$if\ \beta = 1.5 \quad P = z^{-5} f_x^{-1}$$

$$show\ that\ P(z) < P(1) = 1$$

$$look\ up\ f_x\ in\ tables$$

carbon	$z = 6$	$f_6 = 3.3\times10^{-4}$	$P(6) = 0.4$
nitrogen	$z = 7$	$f_7 = 8.3\times10^{-5}$	$P(7) = 0.7$
oxygen	$z = 8$	$f_8 = 6.8\times10^{-4}$	$P(8) = 0.04$
neon	$z = 10$	$f_{10} = 1.2\times10^{-4}$	$P(10) = 0.08$

$$if\ \beta > 1.5 \quad P(z)\ is\ even\ smaller$$