Astro 201 – Radiative Processes – Solution Set 6

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Problem 1. *Saha and Redshift of Recombination*

Rybicki & Lightman problem 9.4, parts (a), (b), and (c)

Assuming that the partition functions are of order unity, and with $\lambda \equiv h/(2\pi mkT)^{1/2}$, Saha equation connecting two successive ionization stages is:

$$\frac{n_{j+1}}{n_j} \sim \frac{e^{-\chi_j/kT}}{n_e\lambda^3}, \text{ or} \tag{1}$$

$$\ln\left(\frac{n_{j+1}}{n_j}\right) \sim -\frac{\chi_j}{kT} + \ln\left(\frac{1}{n_e\lambda^3}\right). \tag{2}$$

(a) The ionization stage passes from j to $j+1$ when $n_{j+1} \sim n_j$, so:

$$\frac{\chi_j}{kT} \sim \ln\left(\frac{1}{n_e\lambda^3}\right) \equiv \gamma \implies kT \sim \frac{\chi_j}{\gamma} \ll \chi_j. \tag{3}$$

(b) The population ratio changes with respect to temperature like:

$$\frac{d\ln(n_{j+1}/n_j)}{d\ln T} \sim \frac{\chi_j}{kT} + 3/2 \quad \text{since } \; n_e\lambda^3 \propto T^{-3/2}. \tag{4}$$

[Note: $d/d(\ln T) = T d/dT$.] The transition temperature is $kT \sim \chi_j/\gamma$ (and $\gamma \gg 3/2$), so the ionization stage changes over a temperature range

$$\Delta T \sim T\left[\frac{d\ln(n_{j+1}/n_j)}{d\ln T}\right]^{-1} \sim T\gamma^{-1} \ll T. \tag{5}$$

(c) For an atom/ion in state j, the ratio of excited to ground state populations is:

$$\frac{n_{i,j}}{n_{0,j}} = \frac{g_{i,j}}{g_{0,j}} e^{-\chi_{i,j}/kT}, \quad \text{where } \chi_{i,j} \text{ is the excitation potential.} \tag{6}$$

Before being ionized, $kT \lesssim \chi_j/\gamma$, so:

$$\frac{n_{i,j}}{n_{0,j}} \lesssim \frac{g_{i,j}}{g_{0,j}} e^{-\gamma\chi_{i,j}/\chi_j}. \tag{7}$$

The excitation potential $\chi_{i,j}$ is of the same order as the ionization potential χ_j (except for very low-lying states), so when the electrons are very nondegenerate (γ large), the atom/ion stays mostly in its ground state ($n_{i,j} \ll n_{0,j}$).

(d) Calculate the redshift, z_{rec}, at which recombination occurred in the early universe. Use the temperature-redshift relation

$$T_{\text{photon}} = T_0(1+z) \tag{8}$$

and the relation between baryonic number density and redshift

$$n = n_0(1+z)^3 \,. \tag{9}$$

Here $T_0 = 2.73\,\text{K}$ is the temperature of the cosmic microwave photon gas today, and $n_0 = 10^{-7}\,\text{cm}^{-3}$ is the number density of baryons (read: hydrogen atoms) today, grossly averaged over the entire universe. You may define the epoch of recombination to be when 0.5 of the protons have recombined to form neutral hydrogen.

(Redshift is a cosmologist's measure of time. A redshift $z = 0$ corresponds to today. As one goes back in time, the redshift increases. Photons that have travelled from an epoch corresponding to a redshift z have their original wavelengths, λ, stretched to a longer wavelength, λ', by the expansion of the universe. By definition, $\lambda'/\lambda = 1+z$.)

(d) Defining the recombination epoch to be when $n_p = n_H$,

$$1 = \frac{n_p}{n_H} = \frac{2U_p(T)}{U_H(T)} \frac{e^{-\chi_H/kT}}{n_e \lambda^3} = \frac{e^{-\chi_H/kT}}{4.1 \times 10^{-16} n_e T^{-3/2}} \text{ (in cgs units)}. \tag{10}$$

For hydrogen in the ground state, $U_H(T) = 2U_p(T)$. With $T = T_0(1+z)$ and $n_e = n_{e,0}(1+z)^3 = n_0(1+z)^3$,

$$1 = \frac{e^{-\chi_H/kT_0(1+z_{rec})}}{4.1 \times 10^{-16} n_0 T_0^{-3/2} (1+z_{rec})^{3/2}} \text{ (in cgs units)}. \tag{11}$$

With $\chi_H = 13.6\,\text{eV}$ and $T_0 = 2.73\,\text{K}$ ($kT_0 = 0.235\,\text{meV}$), using $n_0 = 10^{-7}/\,\text{cm}^3$ gives a solution $z_{rec} \sim 1400$ (actually less significant figures because a gross average was used for n_0).

Problem 2. *The Orion Nebula*

(a) *Sketch the flux density, F_ν, for the Orion Nebula due to its free-free emission from a wavelength of $\lambda = 100\,\mu\text{m}$ to $\lambda = 100\,\text{cm}$. Express F_ν in Janskys (10^{-23} in cgs units; see PS 4 if in doubt). Plot the spectrum on a log-log plot, and indicate power-law indices where appropriate.*

Take the density of electrons to be $n_e = 2 \times 10^3\,\text{cm}^{-3}$, the electron and proton temperatures to be $T = T_e = T_p = 8 \times 10^3\,\text{K}$, the dimension of the ionized cloud to be $R = 1\,\text{pc}$, and the distance to the Orion Nebula to be $d = 500\,\text{pc}$. Take whatever geometry (cube, sphere) for the cloud is most convenient.

Please do not forget free-free self-absorption.

For a Maxwellian (thermal) distribution of particle speeds,

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{\frac{-mv^2}{2kT}}$$

free-free radiation is **emitted** at a rate per volume per frequency of:

$$\frac{dW}{dV\,dt\,d\nu} = \frac{2^5\pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} Z^2 T^{-1/2} e^{-\frac{h\nu}{kT}} \overline{g}_{ff} = \mathcal{E}_\nu^{ff} = 4\pi j_\nu^{ff}$$

Also assume thermal free-free **absorption**, so

$$\alpha_\nu^{ff} = \frac{j_\nu^{ff}}{B_\nu(T)} = \frac{\frac{2^5\pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} Z^2 T^{-1/2} n_e n_i e^{-\frac{h\nu}{kT}} \overline{g}_{ff}}{2h\nu^3/[c^2(e^{\frac{h\nu}{kT}}-1)]} = \frac{2^2 e^6 (2\pi)^{1/2} Z^2 n_e n_i e^{-\frac{h\nu}{kT}} \overline{g}_{ff}(1-e^{-\frac{h\nu}{kT}})}{3mc(3km)^{1/2} T^{1/2} h\nu^3}$$

Assume pure ionized Hydrogen, $Z = 1$ and $n_e = n_i = n$.

$$\alpha_\nu^{ff} = \frac{4e^6}{3mhc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} n^2 (1-e^{-h\nu/kT})\overline{g}_{ff}\nu^{-3}$$

Now, to compare hν (= hc/λ) to kT for the spectral range of interest:

$$\frac{hc}{\lambda_{\text{min}}} \simeq \frac{(7\times10^{-27}\text{erg}\cdot\text{s})(3\times10^8\text{m/s})}{10^{-4}\,\text{m}} \simeq 2\times10^{-14}\text{erg}$$

$$\frac{hc}{\lambda_{\text{max}}} \simeq \frac{(7\times10^{-27}\text{erg}\cdot\text{s})(3\times10^8\text{m/s})}{1\,\text{m}} \simeq 2\times10^{-18}\text{erg}$$

$$kT \simeq (10^{-16}\text{erg/K})(8\times10^3\text{K}) \simeq 10^{-12}\text{erg}$$

So, hν << kT for the whole range. This is the *Rayleigh-Jeans* region of the spectrum. So we can approximate (1-e$^{-h\nu/kT}$) as $\simeq (1-(1-\frac{h\nu}{kT})) = \frac{h\nu}{kT}$. Also assume $\overline{g}_{ff} \sim 1$.

$$\alpha_\nu^{ff} \simeq \frac{4e^6}{3mkc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-3/2} n^2 \nu^{-2}$$

So now we have expressions for both α_ν^{ff} and j_ν^{ff}, which gives S$_\nu^{ff}$ = $\frac{\alpha_\nu^{ff}}{j_\nu^{ff}} = \frac{2kT\nu^2}{c^2}$. With this, we can use the solution to the radiative transfer equation:

$$I_\nu(\tau_\nu) = \int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(\tau_\nu - t_\nu)} dt_\nu + I_\nu(t_\nu = 0) e^{-\tau_\nu}$$

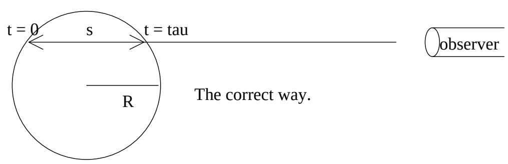
For constant S_ν (assume uniform cloud composition) and $I_\nu(t_\nu = 0) = 0$ (no background lighting), this equation simplifies to

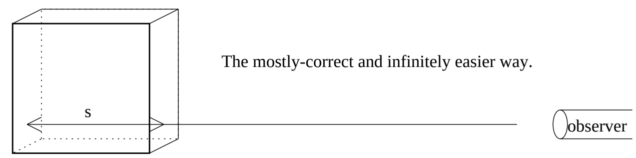
$$I_\nu(\tau_\nu) = S_\nu e^{-\tau_\nu} \int_0^{\tau_\nu} e^{t_\nu} dt_\nu = S_\nu e^{-\tau_\nu} \left[e^{t_\nu}\right]_0^{\tau_\nu} = S_\nu e^{-\tau_\nu}(e^{\tau_\nu} - e^0) = S_\nu(1 - e^{-\tau_\nu})$$

For a uniform medium, $\tau_\nu = \int_0^s \alpha_\nu(s')ds' = \alpha_\nu \cdot s$, where s is the path length.

$$I_\nu^{ff}(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu}) = S_\nu(1 - e^{-\alpha_\nu \cdot s}) = \frac{2kT\nu^2}{c^2}[1 - e^{\frac{-4e^6}{3mkc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-3/2} n^2 \bar{g}_{ff} \nu^{-2} \cdot s}]$$

But what is "s"? Well, to be careful I would need to calculate it for every line of sight through this spherical cloud (see below, left).





But I am far too lazy for that, so I will assume that the Orion nebula is one of an exotic new class of objects discovered by graduate students. Behold the mysterious cubical nebula (above, right). Hey, at least I conserved volume! $s = (\frac{4\pi}{3})^{1/3} R$ ...

Finally, to turn this specific intensity into a flux, we must multiply by the solid angle subtended by the object from our position. $F_\nu = I_\nu d\Omega$. At distance d, $d\Omega = \frac{s^2}{d^2}$. So

$$F_\nu(\tau_\nu) = 2kT(\frac{\nu s}{cd})^2(1 - e^{\frac{-4e^6}{3mkc}(\frac{2\pi}{3km})^{1/2} T^{-3/2} n^2 \nu^{-2} \cdot s}),$$

where d = 500 pc, s = $(\frac{4\pi}{3})^{1/3}$R, R = 1 pc, n = 2000 cm^{-3}, T = 8000 K. ν is the only variable. Everything else is a constant.

The units are $\left[\frac{\text{erg K pc}^2\text{s}^2}{\text{K s}^2\text{cm}^2\text{pc}^2}\right] = \left[\frac{\text{erg}}{\text{cm}^2\cdot\text{s}\cdot\text{Hz}}\right]$. To express in Janskys, divide by 10^{-23}. See the plot a few pages from now. The spectral slope is nearly zero on high frequency portion of the spectrum, and is 2 on the low frequency portion.

(b) *Overlay on your plot F_ν from pure electron-electron scatterings. Here aim only for order-of-magnitude accuracy. Remember that electron-electron collisions involve time-varying quadrupole moments, not time-varying dipole moments, and recall the quadrupole vs. dipole scalings from our discussion of Einstein A's.*

For this part, consider the emissivity (j_ν) purely from electron-electron scatterings. Drop the emissivity from electron-proton scatterings. However, for the absorptivity (α_ν), consider the contributions from both electron-proton and electron-electron scatterings. If you think one absorption process is more important than the other, justify why you think that is the case and proceed by including just the one.

Recall from lecture our discussion of the Einstein coefficients: $\mathrm{A}_{21,\text{electric quadrupole}} \sim \mathrm{A}_{21,\text{electric dipole}} \times \left[\frac{\mathrm{S}}{\lambda}\right]^2$, where λ is the wavelength of the photon created in the interaction, and s is the "characteristic length scale" of the problem, which I will consider as the average impact parameter and call "b". The ratio of Einstein A's is just the ratio of *power* emitted by a dipole vs. *power* radiated by a quadrupole. Since e-/ion is a dipolar and e-/e- is a quadrupolar interaction, the emission coefficients are related like the Einstein A's above:

$$j_\nu^{ee} \sim j_\nu^{ff} \times \left(\frac{b}{\lambda}\right)^2$$

And, because this will still be *thermal* e-/e- absorption as well as emission (just like in the e-/ion case before), the source function is the Planck function, $B_\nu(T_e) = B_\nu(T)$. Rearranging the definition of the source function as the ratio of the emission and absorption coefficients gives a relationship between the e-/e- absorption coefficients that closely resembles that of the e-/ion absorption coefficients:

$$\alpha_\nu^{ee} = \frac{j_\nu^{ee}}{B_\nu(T)} = \frac{j_\nu^{ff}(\frac{b}{\lambda})^2}{B_\nu(T)} = \frac{j_\nu^{ff}}{B_\nu(T)}\left(\frac{b}{\lambda}\right)^2 = \alpha_\nu^{ff} \times \left(\frac{b}{\lambda}\right)^2$$

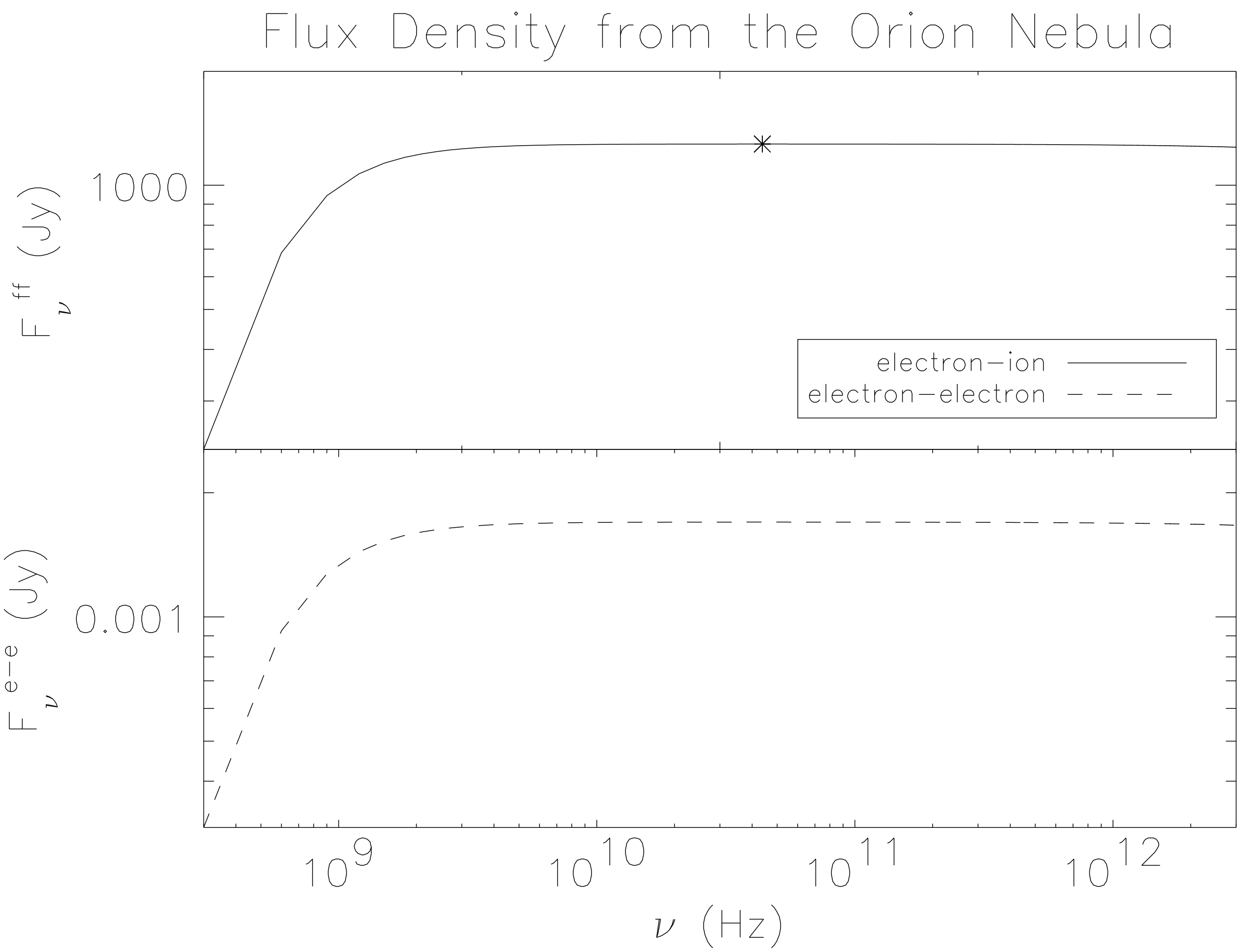
But what is the average impact parameter, b? Recall our derivation of electron-proton bremmstrahlung in which we identified each encounter having impact parameter b as generating radiation having a typical frequency $\nu \sim v/b$, where v was the velocity of encounter. Then

$$\left(\frac{b}{\lambda}\right)^2 \sim \left(\frac{v}{c}\right)^2 \sim \frac{kT}{mc^2} \sim 10^{-6} \tag{12}$$

Does one absorption process—either e-/ion free-free or e-/e- scattering—dominate over the other? Since $(b/\lambda)^2 \sim 10^{-6}$ is very small, e-/ion free-free dominates the absorption. Then we can write

$$I_\nu^{ee} = S_\nu^{ee}(1 - e^{-\tau_\nu}) = \frac{j_\nu^{ee}}{\alpha_\nu^{ee} + \alpha_\nu^{ff}}(1 - e^{-(\alpha_\nu^{ee} + \alpha_\nu^{ff})\cdot s}) \simeq \frac{j_\nu^{ff}(b/\lambda)^2}{\alpha_\nu^{ff}}(1 - e^{-\alpha_\nu^{ff}\cdot s}) = 10^{-6} \times I_\nu^{ff}$$

As before, we multiply by s^2/d^2 to convert to Flux. See the second plot below the first one on the next page.



No wonder nobody ever includes it! The entire e-e spectrum is down by 6 orders of magnitude w.r.t. the usual e-proton spectrum.

Problem 3. *Blowing Stromgren Bubbles*

Consider a lone O star emitting η Lyman limit photons per second. It sits inside hydrogen gas of infinite extent and of number density n. The star ionizes an HII region—a.k.a. a "Stromgren sphere," after Bengt Stromgren, who understood that such spheres have sharp boundaries—in which hydrogen is nearly completely ionized.

Every Lyman limit photon goes towards ionizing a neutral hydrogen atom. That is, every photon emitted by the star goes towards maintaining the Stromgren bubble. Put yet another way, no Lyman limit photon emitted by the star travels past the radius of the Stromgren sphere.

The rate at which Lyman limit photons are emitted by the central star equals the rate of radiative recombinations in the ionized gas. The sphere is nearly completely ionized.[1] *These facts of photo-ionization equilibrium determine the approximate radius of the Stromgren sphere.*

The temperature inside the sphere is about 10000 K. (A class on the interstellar medium can show you why.)

[1] The sphere cannot be 100% ionized because then there would be no neutrals to absorb the Lyman limit photons that are continuously streaming out of the star.

(a) Derive a symbolic expression for the radius of the Stromgren sphere using the above variables and whatever variables were introduced in lecture.

Every Lyman limit photon is used to make a photo-ionization. The rate of photo-ionizations equals the rate of radiative recombinations in steady state. Then

$$\eta = \frac{4}{3}\pi R_s^3 n_p n_e \alpha \tag{13}$$

where R_s is the radius of the Stromgren sphere and α is the recombination coefficient discussed in class. Since the gas is nearly fully ionized, $n_p = n_e = n$, and therefore the radius of the Stromgren sphere is

$$R_s = \left(\frac{3\eta}{4\pi n^2 \alpha}\right)^{1/3} \tag{14}$$

Knowing that every Lyman limit photon is ultimately absorbed by the sphere, we can say that taking $\alpha = \alpha_B$ (on-the-spot approximation) is a better choice for deciding the global size of the sphere.

(b) What is the timescale, t_{rec}, over which a free proton radiatively recombines in the sphere? That is, how long would a free proton have to wait before undergoing a radiative recombination? Give both a symbolic expression, and a numerical evaluation for $n = 1\,\mathrm{cm}^{-3}$.

At photo-ionization equilibrium, the rate η of Lyman limit photon emission equals the rate of radiative recombinations in the sphere (of volume V_s):

$$\eta \simeq \alpha n_p n_e V_s \simeq \alpha n^2 V_s, \tag{15}$$

where α is the net recombination coefficient for hydrogen (taking into account "on-the-spot" re-ionization by photons from radiative recombinations into the ground level). At 10000 K, $\alpha = 2.59 \times 10^{-13}\,\mathrm{cm}^3/\,\mathrm{s}$ (Osterbrock).

A free proton would have to wait before undergoing a radiative recombination about

$$t_{rec} \sim \frac{\text{total \# of protons in the sphere}}{\text{rate of radiative recombinations in the sphere}} \sim \frac{nV_s}{\eta} \sim \frac{1}{\alpha n}. \tag{16}$$

For $n = 1/\,\mathrm{cm}^3$ and at 10000 K, $t_{rec} \sim$ 100,000 years.

(c) If the star were initially "off," and the gas surrounding it initially neutral, what is the timescale for the Stromgren sphere to develop after the star were turned "on"? That is, how long does the star take to blow an ionized bubble? Think simply and to order-of-magnitude; you should get the same answer as (a).

Initially surrounded by neutral gas, the star develops an ionized sphere around it over a timescale roughly equaling (ignoring radiative recombination):

$$\frac{\text{\# of H atoms to ionize to create the Stromgren sphere}}{\text{rate of Lyman limit photon emission}} \sim \frac{nV_s}{\eta} \sim \frac{1}{\alpha n}. \tag{17}$$

Problem 4. *Time to Relax in the Stromgren Sphere*

It is often assumed that velocity distributions of particles are Maxwellian. The validity of this assumption rests on the ability of particles to collide elastically with one another and share their kinetic energy. For a Maxwellian to be appropriate, the timescale for a collision must be short compared to other timescales of interest. This problem tests these assumptions for the case of the Stromgren sphere of nearly completely ionized hydrogen of problem 3.

(a) Establishing the electron (kinetic) temperature: what is the timescale, t_e, for free electrons in the Stromgren sphere to collide with one another? Consider collisions occurring at relative velocities typical of those in an electron gas at temperature T_e. Work only to order-of-magnitude and express your answer in terms of n, T_e, and other fundamental constants.

Establishing the electron (kinetic) temperature: The timescale t_e for free electrons to collide with one another is: $t_e \sim 1/n\sigma v$, where σ is the collisional cross-section. For two charged particles to share their kinetic energy, they must approach each other to within a separation r where their electric potential is comparable to their kinetic energy: $e^2/r \sim mv^2$. So, $\sigma \sim r^2 \sim e^4/(mv^2)^2$. With $mv^2 \sim kT_e$,

$$t_e \sim \frac{\sqrt{m_e}(kT_e)^{3/2}}{ne^4} \sim 10 \text{ days (for } 10^4\,\text{K).} \tag{18}$$

(b) Establishing the proton (kinetic) temperature: repeat (a), but for protons, and consider collisions at relative velocities typical of those in a proton gas of temperature T_p. Call the proton relaxation time t_p.

Establishing the proton (kinetic) temperature: Similarly the proton relaxation time is:

$$t_p \sim \frac{\sqrt{m_p}(kT_p)^{3/2}}{ne^4} \sim \sqrt{\frac{m_p}{m_e}}t_e \sim 1 \text{ year (for } 10^4\,\text{K).} \tag{19}$$

(c) Establishing a common (kinetic) temperature: suppose that initially, $T_e > T_p$. What is the timescale over which electrons and protons equilibrate to a common kinetic temperature? This is not merely the timescale for a proton to collide with an electron. You must consider also the amount of energy exchanged between an electron and proton during each encounter. Estimate, to order-of-magnitude, the time it takes a cold proton to acquire the same kinetic energy as a hot electron. Call this time t_{ep}. Again, express your answer symbolically.

Hint: you might find it helpful to switch the charge on the electron and consider head-on collisions between the positive electron and positive proton.

Establishing a common (kinetic) temperature: Consider collisions in which the electron is significantly deflected by the proton. These occur, to order-of-magnitude, over a timescale t_e. In such collisions, the change in the electron's velocity Δv_e is comparable to the original electron velocity v_e. The proton's momentum changes by $m_p\Delta v_p \sim m_e\Delta v_e$ by momentum conservation. Therefore $\Delta v_p \sim (m_e/m_p)v_e$.

The amount of ENERGY gained by the proton is $\Delta E_p \sim m_p(\Delta v_p)^2$. Each collision imparts ΔE_p and it takes $N \sim m_e v_e^2/\Delta E_p$ number of collisions for the proton's energy to rise up to the electron's energy. Putting it all together, $N \sim m_p/m_e$. Therefore $t_{ep} \sim N t_e \sim (m_p/m_e)t_e$. Numerically,

$$t_{ep} \sim \frac{m_p}{m_e} t_e \sim 60 \text{ years (for } 10^4\,\text{K}). \tag{20}$$

What if instead of considering the energy impulse ΔE_p we considered the velocity impulse Δv_p? Can we reason that the final equilibrium velocity of the proton must be $\sqrt{m_e/m_p} v_e$ and therefore the number of collisions required to reach equilibrium is $\sqrt{m_e/m_p} v_e/\Delta v_p$? The answer is no; velocity has direction and therefore undergoes a random walk. The root-mean-square of the velocity increases not as the number of collisions but rather more slowly as the square root of the number of collisions. Since the velocity random walks, the energy walks (regularly), a result that returns us to the derivation above.

(d) Numerically evaluate t_e/t_{rec}, t_p/t_{rec}, and t_{ep}/t_{rec}, for $T_e \sim T_p \sim 10^4\,\text{K}$. Is assuming a Maxwellian distribution of velocities at a common temperature for both electrons and ions a good approximation in Stromgren spheres?

For $T_e \sim T_p \sim 10^4\,\text{K}$,

$$\frac{t_e}{t_{rec}} \sim \frac{1}{10^{6.5}}, \qquad \frac{t_p}{t_{rec}} \sim \frac{1}{10^5}, \qquad \frac{t_{ep}}{t_{rec}} \sim \frac{1}{2000}. \tag{21}$$

Since the timescales for these collisions are shorter than that of radiative recombinations in Stromgren spheres, the velocity distributions can be approximated Maxwellian at a common temperature for both electrons and protons.