Astro 201 – Radiative Processes – Problem Set 8

Readings: Rybicki & Lightman Chapter 5

Problem 1. Seeing the Forest Through the Clouds

Optical spectra of high redshift quasars often exhibit a dense thicket of absorption lines (see examples from Wolfe et al. 1993; ApJ, 404, 480). Many (but not all) of these absorption features are thought to be from the same transition: Lyman α ($n = 1$ to $n = 2$) in hydrogen. The absorption lines are located at different wavelengths because they arise from different hydrogen clouds located at different redshifts between us and the quasar (for the latter, read: convenient, bright, continuum light source). This so-called "Lyman α forest" of absorption lines is diagnostic of the degree of clumpiness in gas at high redshift; i.e., it is diagnostic of structure formation.

In this problem, we relate the observed width of each line to the column density of hydrogen in the corresponding "Lyman α cloud." Most of this problem can be done in ignorance of cosmology.

Take the Lyα line profile presented by a single cloud to be both Doppler broadened and naturally broadened. As described by Rybicki & Lightman page 291, the convolved line profile is described by the Voigt function.

(a) If the "a"-parameter in the Voigt function is much less than one, the Voigt function can be described in two parts. Define $u \equiv (\nu - \nu_0)/\Delta\nu_D$, where I am using the notation of Rybicki & Lightman. If $u \ll 1 + a^{-2}$, then

$$\phi(u \ll 1 + a^{-2}) \approx \frac{1}{\sqrt{\pi}\Delta\nu_D} e^{-u^2}\,, \tag{1}$$

while if $u \gg 1 + a^{-2}$, then

$$\phi(u \gg 1 + a^{-2}) \approx \frac{a}{\pi \Delta\nu_D u^2}\,. \tag{2}$$

In other words, the core of the line, near line center, is dominated by the Doppler Gaussian profile, while the wings of the line, far from line center, are dominated by the Lorentzian profile. You can verify that these expressions are nothing more than (10.68) and (10.73) taken in the appropriate limits.

Calculate "a" for the Lyα line and verify that it is much less than one. Use a Doppler width of $\Delta\nu_D = (10\,\mathrm{km/\,s})\nu_0/c$.

(b) The *equivalent width* of a spectral absorption line is, by definition,

$$EW \equiv \int_0^\infty \frac{F_{\lambda,0} - F_\lambda}{F_{\lambda,0}} d\lambda \tag{3}$$

where F_λ is the actual flux density, and $F_{\lambda,0}$ is the flux density of the spectrum *if the absorption line were not present* (i.e., the "continuum" flux density.)

Measurements of the equivalent widths of spectral lines can be used to infer column densities of intervening material.

First suppose that the Lyα cloud is optically thin in the line, so that the line profile looks Gaussian (we cannot probe the Lorentzian wings of the line). Derive a symbolic expression between EW and the column density, N_{HI}, of neutral hydrogen in the cloud, using whatever fundamental constants you need.

Hint: The frequency width of the line is still narrow compared to the line frequency (the line is not relativistically broadened). Approximate integrals in light of this fact.

(c) Now suppose that the Lyα cloud is so optically thick in the line that we can make out the naturally broadened Lorentzian wings of the line. That is, the core of the line is completely black, but far from line center, the flux rises back up according to the quasi-power-law Lorentzian. Derive a symbolic expression between EW and N_{HI} in this regime.

Hint: Using the Lorentzian function in the core is a permitted mathematical convenience because the core is completely blackened.

(d) Clouds in regime (b) are referred to as members of the "Lyα forest." Clouds in regime (c) are referred to as "damped Lyα systems"—the "damped" refers to the fact that we are sensitive to the "naturally damped" Lorentzian wings of the line.

In Wolfe et al.'s paper, we can distinguish Lyα forest clouds having observed $EW_{obs} \sim 0.2\mathring{A}$. Numerically estimate N_{HI} for such clouds, assuming they are located at redshift $z = 3$.

Remember that spectral line profiles get stretched with the expansion of the universe, so $EW_{obs} = (1+z)EW_{int}$, where EW_{int} is the intrinsic equivalent width of the line (the EW you would measure if you were positioned right behind the cloud). All of your expressions for (a)–(c) pertain to EW_{int}.

(e) In the same quasar spectrum, we can also distinguish damped Lyα systems having $EW_{obs} \sim 100\mathring{A}$. Numerically estimate N_{HI} for such clouds, and compare your answer to the column in our Galaxy, $N_{HI,Gal} \sim 10^{21}\,\mathrm{cm}^{-2}$. The fact that they are comparable argues that damped Lyα systems are full-fledged galaxies that happen to lie between us and the quasar.

Problem 2. Protogalaxies

Why do galaxies have the sizes and masses that they do?

Here is a rule of thumb governing the sizes of objects that can collapse under the weight of their own self-gravity: objects can only collapse if their *cooling time* is shorter than their *gravitational collapse time.* The cooling time of gas is the time it takes to lose order unity (say, 0.5) of its thermal energy. The collapse time is the time it takes to shrink its radius by order unity. (This rule of thumb is subject to details regarding the exact cooling mechanisms; i.e., the effective adiabatic index of collapsing material. The adiabatic index doesn't have to be exactly one (isothermal) for collapse to proceed.)

(a) Present an order-of-magnitude derivation, using only bremsstrahlung radiation to cool off gas, of the *maximum* radius R of an object that can form out of a *virialized* lump of hydrogen plasma. (Recall that if a gas is virialized, its kinetic temperature is uniquely related to the depth of its gravitational potential well. Gas becomes virialized as it collapses under its own self-gravity and shocks; the kinetic energy of collapse is converted efficiently into heat in these shocks.)

Express in kpc.

(b) For an object having such a maximum radius R, present an order-of-magnitude derivation of its *minimum* mass M by considering how hot the gas must be to be a fully ionized plasma.

Express in solar masses.

OPTIONAL Problem 3. Sobolev Escape

Imagine a homogeneous sphere of uniform density that emits and absorbs photons of a certain frequency. Every volume element of the sphere produces a certain number of photons per second, and these photons are unleashed from each volume element isotropically. The material is purely absorbing; neglect scattering.

The radial optical depth of the sphere to these photons is τ (integrated over the radius of the sphere).

(a) What fraction of the photons that are unleashed per second (throughout the entire sphere) actually escape the sphere per second (unabsorbed)? Express in terms of τ. You have calculated what is called the "escape probability for a uniform absorbing sphere," useful for more technical analyses of radiative transfer.

(b) Explain why your answer in (a) makes sense in the limits that $\tau \ll 1$ and $\tau \gg 1$.

You may find it helpful to solve (a) to order-to-magnitude in these limits before trying to attack (a) in full generality. This problem can, with a modicum of effort, be solved exactly.