Astro 201 – Radiative Processes – Solution Set 7

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Readings: Rybicki & Lightman Chapter 5

Problem 1. *Seeing the Forest Through the Clouds*

Optical spectra of high redshift quasars often exhibit a dense thicket of absorption lines (see examples from Wolfe et al. 1993; ApJ, 404, 480). Many (but not all) of these absorption features are thought to be from the same transition: Lyman α ($n = 1$ to $n = 2$) in hydrogen. The absorption lines are located at different wavelengths because they arise from different hydrogen clouds located at different redshifts between us and the quasar (for the latter, read: convenient, bright, continuum light source). This so-called "Lyman α forest" of absorption lines is diagnostic of the degree of clumpiness in gas at high redshift; i.e., it is diagnostic of structure formation.

In this problem, we relate the observed width of each line to the column density of hydrogen in the corresponding "Lyman α cloud." Most of this problem can be done in ignorance of cosmology.

Take the Lyα line profile presented by a single cloud to be both Doppler broadened and naturally broadened. As described by Rybicki & Lightman page 291, the convolved line profile is described by the Voigt function.

(a) If the "a"-parameter in the Voigt function is much less than one, the Voigt function can be described in two parts. Define $u \equiv (\nu - \nu_0)/\Delta\nu_D$, where I am using the notation of Rybicki & Lightman. If $u \ll 1 + a^{-2}$, then

$$\phi(u \ll 1 + a^{-2}) \approx \frac{1}{\sqrt{\pi}\Delta\nu_D} e^{-u^2} \,, \tag{1}$$

while if $u \gg 1 + a^{-2}$, then

$$\phi(u \gg 1 + a^{-2}) \approx \frac{a}{\pi \Delta\nu_D u^2} \,. \tag{2}$$

In other words, the core of the line, near line center, is dominated by the Doppler Gaussian profile, while the wings of the line, far from line center, are dominated by the Lorentzian profile. You can verify that these expressions are nothing more than (10.68) and (10.73) taken in the appropriate limits.

Calculate "a" for the Lyα line and verify that it is much less than one. Use a Doppler width of $\Delta\nu_D = (10\,\mathrm{km/s})\nu_0/c$.

For the Lyα line that is Doppler broadened [by width $\Delta\nu_D = (10\,\text{km/s})\nu_0/c = 8\times10^{10}$ Hz for $\lambda_0 = 1216\mathring{A}$], and naturally broadened [$\Gamma = A_{21} = 6 \times 10^8$ Hz],

$$a \equiv \frac{\Gamma}{4\pi\Delta\nu_D} \approx 6 \times 10^{-4} \ll 1. \tag{3}$$

(b) The equivalent width of a spectral absorption line is, by definition,

$$EW \equiv \int_0^\infty \frac{F_{\lambda,0} - F_\lambda}{F_{\lambda,0}} d\lambda \tag{4}$$

where F_λ is the actual flux density, and $F_{\lambda,0}$ is the flux density of the spectrum if the absorption line were not present (i.e., the "continuum" flux density.)

Measurements of the equivalent widths of spectral lines can be used to infer column densities of intervening material.

First suppose that the Lyα cloud is optically thin in the line, so that the line profile looks Gaussian (we cannot probe the Lorentzian wings of the line). Derive a symbolic expression between EW and the column density, N_{HI}, of neutral hydrogen in the cloud, using whatever fundamental constants you need.

Assuming that the hydrogen is purely absorbing, $F_\lambda = F_{\lambda,0}e^{-\tau(\lambda)}$, where $\tau(\lambda)$ is the optical depth to absorption. Then the equivalent width of a spectral absorption line is:

$$EW \equiv \int_0^\infty \frac{F_{\lambda,0} - F_\lambda}{F_{\lambda,0}} d\lambda = \int_0^\infty 1 - e^{-\tau(\lambda)} d\lambda. \tag{5}$$

The optical depth depends on wavelength through the absorption cross-section $\sigma(\lambda)$:

$$\tau(\lambda) = N_{\text{HI}}\sigma(\lambda) = N_{\text{HI}}\frac{A_{21}}{8\pi}\lambda_0^2\phi(\nu). \tag{6}$$

If the Lyα cloud is optically thin in the line ($\tau \ll 1$),

$$EW_{thin} \simeq \int_0^\infty \tau(\lambda)d\lambda \simeq N_{\text{HI}}\frac{A_{21}\lambda_0^2}{8\pi}\int_0^\infty \phi(\nu)d\lambda, \simeq N_{\text{HI}}\frac{A_{21}\lambda_0^2 c}{8\pi}\int_0^\infty \frac{\phi(\nu)}{\nu^2}d\nu, \tag{7}$$

and the line profile looks Gaussian: $\phi(u) \approx \frac{1}{\sqrt{\pi}\Delta\nu_D}e^{-u^2}$, where $u \equiv (\nu - \nu_0)/\Delta\nu_D$. Now $\phi(\nu)$ is sufficiently narrow that the ν^2-denominator in the above integral hardly changes, so we can take the ν^2 outside as ν_0^2:

$$EW_{thin} \simeq N_{\rm HI} \frac{A_{21}\lambda_0^2 c}{8\pi\nu_0^2} \int_0^\infty \phi(\nu) d\nu = N_{\rm HI} \frac{A_{21}\lambda_0^2 c}{8\pi\nu_0^2} = N_{\rm HI} \frac{A_{21}\lambda_0^4}{8\pi c}. \tag{8}$$

This result is referred to as the "linear portion of the curve of growth."

(c) Now suppose that the Lyα cloud is so optically thick in the line that we can make out the naturally broadened Lorentzian wings of the line. That is, the core of the line is completely black, but far from line center, the flux rises back up according to the quasi-power-law Lorentzian. Derive a symbolic expression between EW and N_{HI} in this regime.

If the Lyα cloud is so optically thick in the line that the core is completely black, but the flux rises back up in the naturally broadened wings of the line ($\phi(u) \approx \frac{a}{\pi \Delta\nu_D u^2}$):

$$\tau = N_{\rm HI} \frac{A_{21}}{8\pi} \lambda_0^2 \phi(\nu) \approx N_{\rm HI} \frac{a A_{21}}{8\pi^2 \Delta\nu_D u^2} \lambda_0^2 \tag{9}$$

Notice we have pulled a trick here: we used only the asymptotic Lorentzian shape of the line, even though we know the line shape in the core is Gaussian, not Lorentzian. We can do this because the line core is pitch black, a feature that the Lorentzian shape captures just fine—after all, using the Lorentzian at $u = 0$, we get infinite optical depth!

Letting $\alpha \equiv a A_{21} \lambda_0^2 / (8\pi^2 \Delta\nu_D)$

$$EW_{thick} \simeq c \int_0^\infty \frac{1 - e^{-\alpha N_{\rm HI}/u^2}}{\nu^2} d\nu \tag{10}$$

Again, the integral is dominated by small u, near line center. Provided the line width in frequency is still smaller than the frequency (the line is not relativistically broad), we can again take the ν^2-denominator outside as ν_0^2:

$$EW_{thick} \simeq \frac{c}{\nu_0^2} \int_0^\infty 1 - e^{-\alpha N_{\rm HI}/u^2} d\nu \tag{11}$$

Now it is a matter of non-dimensionalizing the integral. Note that

$$\nu = \nu_0 + u\Delta\nu_D = \nu_0(1 + u\beta), \text{ where } \beta \equiv \frac{\Delta\nu_D}{\nu_0} = \frac{10\,\rm km/s}{c}, \tag{12}$$

to write the above integral as

$$\Delta\nu_D \frac{c}{\nu_0^2} \int_{-1/\beta}^\infty 1 - e^{-\alpha N_{\rm HI}/u^2} du \tag{13}$$

Make a change of variable to x such that $x^{-2} = u^{-2}\alpha N_{\rm HI}$. Then the integral reads

$$\Delta\nu_D \frac{c}{\nu_0^2}\sqrt{\alpha N_{\rm HI}} \int_{-1/(\beta\sqrt{\alpha N_{\rm HI}})}^{\infty} 1 - e^{-1/x^2} dx \tag{14}$$

The lower limit on the integral is $\ll -1$, as can be verified for any particular problem (we can verify it for the cases below, if we wish). Then the integral is insensitive to the value of $N_{\rm HI}$ and is approximately $2\sqrt{\pi}$. We conclude that

$$EW_{thick} \approx \sqrt{N_{\rm HI}} \frac{A_{21}\lambda_0^3}{\sqrt{8\pi c}}. \tag{15}$$

This is referred to as the "square root portion of the curve of growth." We can derive roughly the same result more simply by saying that the line is black all the way from line center until the frequency where $\tau = 1$ (say; you could choose another favorite number > 1). Use the approximate Lorentzian form of the line profile function to decide at what frequency $\tau = 1$. Then take the equivalent width to be the area in the black square well of the line, ignoring the wings of the line. This procedure will yield about the same result as above.

(d) Clouds in regime (b) are referred to as members of the "Lyα forest." Clouds in regime (c) are referred to as "damped Lyα systems"—the "damped" refers to the fact that we are sensitive to the "naturally damped" Lorentzian wings of the line.

In Wolfe et al.'s paper, we can distinguish Lyα forest clouds having observed $EW_{obs} \sim 1Å$. Numerically estimate N_{HI} for such clouds, assuming they are located at redshift $z = 3$.

Remember that spectral line profiles get stretched with the expansion of the universe, so $EW_{obs} = (1+z)EW_{int}$, where EW_{int} is the intrinsic equivalent width of the line (the EW you would measure if you were positioned right behind the cloud). All of your expressions for (a)–(c) pertain to EW_{int}.

Assuming the Lyα forest clouds are at redshift $z = 3$, the intrinsic $EW_{thin} = EW_{obs}/4 \sim 0.25Å$.

$$N_{{\rm HI}thin} \approx \frac{8\pi c}{A_{21}\lambda_0^4} EW_{thin} \sim 1.4 \times 10^{14}/\,{\rm cm}^2. \tag{16}$$

We can check *a posteriori* whether our assumption that the cloud is optically thin is valid or not. For the above value of $N \approx 10^{14}\,{\rm cm}^{-2}$, $\tau(\nu_0) \approx 3$, which means that our assumption that this cloud is optically thin is marginally violated.

(e) In the same quasar spectrum, we can also distinguish damped Lyα systems having $EW_{obs} \sim 100Å$. Numerically estimate N_{HI} for such clouds, and compare your answer

to the column in our Galaxy, $N_{HI,Gal} \sim 10^{21}\,\mathrm{cm}^{-2}$. The fact that they are comparable argues that damped Lyα systems are full-fledged galaxies that happen to lie between us and the quasar.

Assuming the damped Lyα systems are also at redshift $z = 3$, the intrinsic $EW_{thick} = EW_{obs}/4 \sim 25\AA$.

$$N_{\mathrm{HI}thick} \approx \left(\frac{\sqrt{8}\pi c}{A_{21}\lambda_0^3} EW_{thick}\right)^2 \sim 4 \times 10^{21}/\,\mathrm{cm}^2, \tag{17}$$

comparable to the column density in our Galaxy of $\sim 10^{21}/\,\mathrm{cm}^2$.

Problem 2. *Protogalaxies*

Why do galaxies have the sizes and masses that they do?

Here is a rule of thumb governing the sizes of objects that can collapse under the weight of their own self-gravity: objects can only collapse if their cooling time is shorter than their gravitational collapse time. The cooling time of gas is the time it takes to lose order unity (say, 0.5) of its thermal energy. The collapse time is the time it takes to shrink its radius by order unity. (This rule of thumb is subject to details regarding the exact cooling mechanisms; i.e., the effective adiabatic index of collapsing material. Truth be told, the adiabatic index doesn't have to be exactly one (isothermal) for collapse to proceed.)

(a) Present an order-of-magnitude derivation, using only bremsstrahlung radiation to cool off gas, of the maximum radius R of an object that can form out of a virialized lump of hydrogen plasma. (Recall that if a gas is virialized, its kinetic temperature is uniquely related to the depth of its gravitational potential well. Gas becomes virialized as it collapses under its own self-gravity and shocks; the kinetic energy of collapse is converted efficiently into heat in these shocks.)

Express in kpc.

By the virial theorem, 2 K.E. = -P.E.$_{grav}$, so

$$2(\frac{3}{2}NkT) = \frac{GM^2}{R} \Rightarrow T(t) = \frac{GM^2}{3NkR(t)} = \frac{GMm_H}{3kR(t)}$$

Now investigating how long it takes the gas to cool:

$$t_{cool} \sim \frac{\text{amount of energy}}{\text{rate of energy radiation}} \sim \frac{1}{2}(\frac{E_0}{dE/dt}) = \frac{1}{2}(\frac{E_0}{\frac{dE}{dt \cdot dVol} \cdot Vol.}) \simeq \frac{1}{2}(\frac{\frac{3}{2}NkT_0}{j \cdot Vol})$$

Note this assumes that the cloud is optically thin. Further assuming that Bremstrahlung is the only cooling mechanism, I use $dW/(dV \times dt)$, the frequency-INTEGRATED emission coefficient from Rybicki & Lightman (5.25) (or 5.15b would have done as well). I assume $\overline{g}_B \sim 1$, $Z \simeq 1$, and $n_e = n_i = n$, and substitute $N/Vol = n = \frac{M}{Vol \cdot m_H}$.

$$t_{cool} \sim \frac{1}{2}\left[\frac{\frac{3}{2}nkT_0}{1.4 \times 10^{-27}(\frac{\text{erg cm}^3}{\text{s K}^{1/2}})T^{1/2}n^2}\right] \simeq \frac{.5 \times 10^{27}(\frac{\text{s K}^{1/2}}{\text{erg cm}^3})kT^{1/2}}{n} = \frac{2 \times 10^{27}(\frac{\text{s K}^{1/2}}{\text{erg cm}^3})kT^{1/2}R^3 m_H}{M}$$

Now investigating the collapse time using the very crude assumption that the gas is in free-fall, so acceleration $a = g$. Taking advantage of the spirit of order of magnitude, I will do dimensional analysis derivation:

$$a = g \Rightarrow \frac{d^2r}{dt^2} = \frac{GM}{r^2} \Rightarrow \frac{R^3}{t^2} \sim GM \Rightarrow t_{collapse} \sim (\frac{R^3}{GM})^{1/2}$$

Now enforcing the condition that $t_{cool} < t_{collapse}$,

$$\frac{2 \times 10^{27}(\frac{\text{s K}^{1/2}}{\text{erg cm}^3})kT^{1/2}R^3 m_H}{M} < (\frac{R^3}{GM})^{1/2}$$

Squaring both sides and using the definition of T(t) from the virial theorem above,

$$\frac{4 \times 10^{54}(\frac{\text{K s}^2}{\text{erg}^6\text{erg}^2})k^2R^6m_H^2}{M^2}(\frac{GMm_H}{3kR}) < (\frac{R^3}{GM}) \Rightarrow R^2 < \frac{10^{-54}(\frac{\text{erg}^2\text{cm}^6}{\text{K s}^2})}{m_H^3G^2k}$$

Plugging in all the constants and converting to pc, we find that $R < 6 \times 10^{23}$ cm = 200 kpc. What an eminently reasonable spacing between neighboring galaxies!

(b) For an object having such a maximum radius R, present an order-of-magnitude derivation of its minimum mass M by considering how hot the gas must be to be a fully ionized plasma.

Express in solar masses.

Getting the mass limit is a little trickier, because M cancels out of the $t_{cool} < t_{collapse}$ condition above, so we cannot solve for it. (Indeed, we were only able to solve for R in the first place because M cancelled out!) So, I will investigate the assumptions that went into this calculation and see what falls out.

Assumption #1: Cloud is optically thin to Bremstrahlung: $\tau \ll 1$. With a little help from R.&L. (5.20) for a frequency-averaged free-free absorption coefficient, we have

$$\tau = \alpha_\nu^{ff} \cdot R \simeq 1.7 \times 10^{-25} \left[\mathrm{K}^{7/2}\mathrm{cm}^5\right] T^{-7/2} n^2 \cdot R \ll 1$$

$$1.7 \times 10^{-25} \left[\mathrm{K}^{7/2}\mathrm{cm}^5\right] T^{-7/2} (\frac{3M}{4\pi m_H R^3})^2 \cdot R \ll 1$$

Substituting in T as a function of M and R from the virial theorem leaves an expression with only M and R as variables.

$$1.7 \times 10^{-25} \left[\mathrm{K}^{7/2}\mathrm{cm}^5\right] (\frac{3kR}{GMm_H})^{7/2} M^2 < 16 m_H^2 R^5$$

Solving for M, and then plugging in constants and the R we just derived gives

$$M > \frac{(3k)^{7/3}(1.7 \times 10^{-25})^{2/3} \left[\mathrm{K}^{7/3}\mathrm{cm}^{10/3}\right])}{16^{2/3} m_H^{11/3} G^{7/3} R} \simeq 9 \times 10^{26} \text{ g} \simeq 4 \times 10^{-7} \text{ solar masses.}$$

Um, I'm gonna be bold and say this is not a terribly interesting lower mass limit.

Assumption #2: Cloud is all ionized Hydrogen. *How hot does it need to be to be ionized?* Well, we know from Saha's equation of ionization equilibrium that the average photon energy should be larger than the binding energy of H's electron, divided by a logarithmic phase space factor (we looked at this in a previous problem set!)

$$h\nu_{peak} > 13.6 \text{ eV}/\chi$$

where $\chi \sim 10$ is the order of magnitude of all logarithms in the universe (this is known as Fermi's law of logarithms). And, using the fact that the Bremstrahlung spectrum peaks at $\nu \sim kT/h$, this becomes a limit on Temperature:

$$h(\frac{kT}{h}) > \frac{13.6 \text{ eV}}{\chi} \Rightarrow T > \frac{13.6 \text{ eV}}{\chi k} \sim \frac{13.6 \text{ eV}}{10 \times 9 \times 10^{-5} \text{ eV/K}} \sim 1 \times 10^4 \text{ K}$$

Now, going back to the virial relation armed with $R \sim 200$ kpc $= 6 \times 10^{23}$ cm and a new lower bound on Temperature,

$$M = \frac{3kTR}{Gm_H} > \frac{3k(1 \times 10^4 \text{ K})(6 \times 10^{23} \text{ cm})}{Gm_H} \sim 2 \times 10^{43} \text{ g} \sim 10^{10} M_\odot \text{ (Solar Masses)}$$

Now this certainly seems like a more interesting lower limit on mass! It's within striking distance of the mass of a Galaxy!! Ah, bliss... This is why I can't stop taking classes.

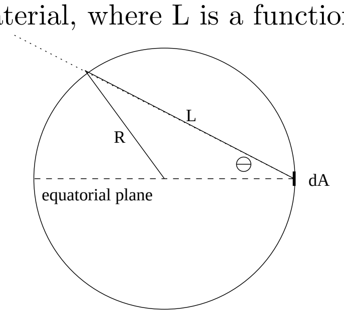
Problem 3. *ESCAPE: A New Formalism by Sobolev*

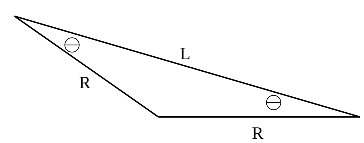
Imagine a homogeneous sphere of uniform density that emits and absorbs photons of a certain frequency. Every volume element of the sphere produces a certain number of photons per second, and these photons are unleashed from each volume element isotropically. The material is purely absorbing; neglect scattering.

The radial optical depth of the sphere to these photons is τ (integrated over the radius of the sphere).

(a) What fraction of the photons that are unleashed per second (throughout the entire sphere) actually escape the sphere per second (unabsorbed)? Express in terms of τ. You have calculated what is called the "escape probability for a uniform absorbing sphere," useful for more technical analyses of radiative transfer.

Consider one small patch on the surface of the gas sphere. The area of the patch is dA. Choose one arbitrary line of sight (L.O.S.) through the sphere starting at dA, with angle θ to the equatorial plane (which contains dA). This L.O.S. passes through length L of cloud material, where L is a function of θ. See figure below left.





The Law of Cosines gives the path length, L, in terms of the cloud radius (R) and θ. See the figure above right.

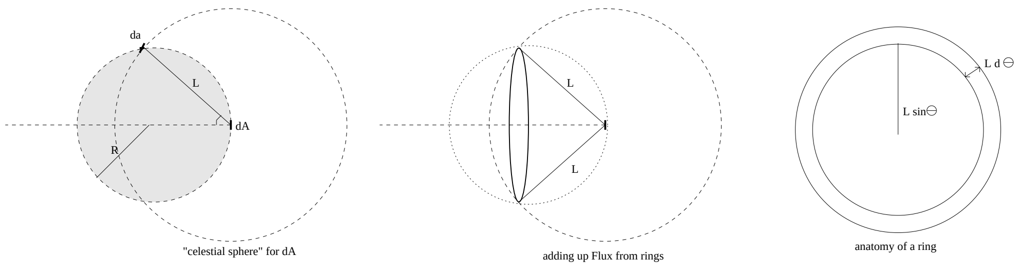
$$R^2 + L^2 - 2RLcos(\theta) = R^2 \Rightarrow L(\theta) = 2Rcos(\theta)$$

Assuming a uniform density inside the sphere and vacuum outside, the source function is constant and there is no intensity entering from outside. τ is defined as $\tau = \alpha R$, so

$$I(\theta) = S(1 - e^{-\tau(\theta)}) = S(1 - e^{-\alpha L(\theta)}) = S(1 - e^{-2R\alpha cos(\theta)}) = \mathbf{S(1 - e^{-2\tau cos(\theta)})}$$

Now, to turn this into a FLUX through dA, consider what the patch dA "SEES" looking into the sphere. dA sees her own celestial sphere of sorts, with herself at the center. Let's say she chooses one particular L.O.S. and its corresponding angle and path length, θ and

L(θ). We'll call the emitting patch at the end of that L.O.S. "da". See below left. To turn the specific intensity of this *emitting* patch da into Flux, we simply multiply by the solid angle subtended by the emitting patch at the point of interest: $d\Omega = cos(\theta)da/L^2$.



Now let us consider **all** such emitting patches 'da' which lie at the same distance L from dA. They lie in a continuous ring along the intersection of the 2 spheres (depicted above middle), and the lines of sight to them describe a cone with opening angle 2θ and vertex at dA. The solid angle subtended by this emitting **ring** (figure above right) is

$$d\Omega_{ring} = \frac{da_{ring}}{L^2}cos(\theta) = \frac{2\pi(Lsin(\theta))Ld\theta}{L^2}cos(\theta) = \frac{2\pi L^2 sin(\theta)d\theta}{L^2}cos(\theta) = 2\pi sin(\theta)cos(\theta)d\theta$$

Now to get the TOTAL flux through dA from *all rings*, we integrate over θ.

$$F_{out} = \int_0^{\pi/2} I(\theta)d\Omega = \int_0^{\pi/2} S(1-e^{-2\tau cos(\theta)})2\pi sin(\theta)cos(\theta)d\theta = 2\pi S\int_0^{\pi/2}(1-e^{-2\tau cos(\theta)})sin(\theta)cos(\theta)d\theta$$

Now substituting $x = 2\tau cos(\theta)$ and writing the source function $S = \frac{j}{\alpha} = \frac{jR}{\tau}$,

$$F_{out} = 2\pi\frac{jR}{\tau}\int_{2\tau}^{0}(1-e^{-x})\frac{x}{2\tau}\frac{dx}{-2\tau}cos(\theta)d\theta = \frac{\pi jR}{2\tau^3}\int_0^{2\tau}(1-e^{-x})xdx = \frac{\pi jR}{2\tau^3}\left[\int_0^{2\tau}xdx + \int_0^{2\tau}-e^{-x}xdx\right]$$

$$F_{out} = \frac{dE}{dtd\nu dA} = \frac{\pi jR}{2\tau^3}\left[2\tau^2 + 2\tau e^{-2\tau} - 1 + e^{-2\tau}\right] \quad (18)$$

This is a flux: the rate that energy escapes *per unit area* per frequency.

To calculate at what rate energy is "unleashed" inside the sphere, I use the volume emissivity, j, remembering that it has units of energy per time per volume per solid angle per frequency: $j = \frac{dE}{dtd(Vol)d\Omega d\nu}$. To get the total energy created in the whole sphere I multiply by its volume. To account for it radiating out isotropically in all directions I multiply by 4π steradians. To turn this into a surface flux, I divide by the surface area of the sphere.

$$F_{tot} = \frac{dE}{dtd\nu dA} = j \times \frac{Vol \times Sol.Angle}{Surf.Area} = j \times \frac{\frac{4}{3}\pi R^3 \times 4\pi}{4\pi R^2} = \frac{4\pi j R}{3}$$

This is the flux of photons that would pass the sphere's surface if there were no absorption or scattering. Now, taking the ratio of the two,

$$\frac{F_{out}}{F_{tot}} = \frac{\frac{\pi j R}{2\tau^3}\left[2\tau^2 + 2\tau e^{-2\tau} - 1 + e^{-2\tau}\right]}{\frac{4\pi j R}{3}} = \mathbf{\frac{3}{8\tau^3}\left[2\tau^2 + 2\tau e^{-2\tau} + e^{-2\tau} - 1\right]} \tag{19}$$

(b) Explain why your answer in (a) makes sense in the limits that $\tau \ll 1$ and $\tau \gg 1$.

You may find it helpful to solve (a) to order-to-magnitude in these limits before trying to attack (a) in full generality.

Now, evaluating this expression in its 2 extreme limits: very large and very small τ.

For $\tau \gg 1$, I expect that no photons will get out. They should all be trapped if the sphere is infinitely thick, so the fraction should go to zero.

For $\tau \to \infty$, $\frac{3}{8\tau^3}\left[2\tau^2 + 2\tau e^{-2\tau} + e^{-2\tau} - 1\right] \simeq \frac{3}{8\tau^3}\left[2\tau^2 - 1\right] \simeq \frac{3}{8\tau^3} \cdot 2\tau^2 = \frac{6}{8\tau} \simeq \frac{3}{4\tau} \to 0.$

Indeed it goes to zero. But there is a bit more information to be interpreted. It goes to zero roughly as $1/\tau$. This makes sense because in an optically thick sphere, the photons that escape are those photons that are emitted within optical depth unity of the surface. All the other photons are unleashed deep in the sphere, and there they are releashed (buried). Therefore the fraction of photons that escape equals the volume of a shell of optical depth unity, divided by the volume of the entire sphere. This equals $4\pi R^2 \Delta R/[(4\pi/3)R^3] = 3\Delta R/R$. But $\alpha R = \tau$ and $\alpha \Delta R \sim 1$. Then we conclude, to order-of-magnitude, that the escape probability equals $3/\tau$, which matches our exact answer to order-of-magnitude. (With hindsight, we could choose $\alpha \Delta R = 1/4$ to get the answers to match exactly).

For $\tau \ll 1$, I expect that all of the photons will escape, so the fraction should go to 1. Now both denominator and numerator go to zero in (19). So we use L'Hopital's rule and differentiate top and bottom, getting

For $\tau \to 0$, $\dfrac{F_{out}}{F_{tot}} \to \dfrac{3}{24\tau^2}\left[4\tau + 2e^{-2\tau} - 4\tau e^{-2\tau} - 2e^{-2\tau}\right] = \dfrac{3}{24\tau^2}\left[4\tau - 4\tau e^{-2\tau}\right]$

which still runs into problems, so we apply the rule again and get

$$\frac{F_{out}}{F_{tot}} \to \frac{3}{48\tau}\left[4 - 4e^{-2\tau} + 8\tau e^{-2\tau}\right]$$

which still runs into problems, so we apply the rule again and get

$$\frac{F_{out}}{F_{tot}} \to \frac{3}{48}\left[8e^{-2\tau} + 8e^{-2\tau} - 16\tau e^{-2\tau}\right] = e^{-2\tau} - \tau e^{-2\tau} \to 1 - 3\tau$$

which most certainly goes to 1 as $\tau \to 0$ (and it is always less than one, as it had better be!)