Astro 201 – Radiative Processes – Solution Set 10

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NOTE FOR ALL PROBLEMS: In this class, we are not going to stress over the $\sin\alpha$ term in any expression involving synchrotron emission, nor are we going to worry about other factors of order unity, like gamma (Γ) functions. If you see a $\sin\alpha$ or gamma function in your travels, just set it equal to 1.

Problem 1. *Leaving on a Jet Frame*[1]

Recall the images of relativistic jets shown in class, and how they appeared brighter on one end than the other.

Consider a bipolar (two-sided), symmetric jet of material moving at BULK velocity v and having BULK Lorentz factor γ. In the rest frame of the jet, the jet axis points at an angle θ relative to our line-of-sight. One side of the jet is directed towards us, and the other side is directed away.

The jet is composed of electrons (possibly also positrons; the situation seems unclear) that radiate synchrotron emission. In the rest frame of the jet, the electrons are still gyrating relativistically, and in that same rest frame, they emit a specific intensity $I_\nu \propto \nu^\alpha$ between frequencies ν_1 and ν_2. For this problem, take $\alpha < 0$.

(a) Sketch the specific intensity of the jet that points toward an observer on Earth, as seen by that observer.

Overlay on this sketch the specific intensity of the counter-jet (again, as seen by the Earth-bound observer).

Overlay also the specific intensity of either jet or counter-jet as measured in its rest frame. Compare in particular the rest-frame jet spectrum with the counter-jet spectrum observed at Earth.

As usual, annotate your sketch with scales and indices, expressed symbolically in terms of the variables given above.

For the jet pointing towards us: the spectrum in the jet rest frame is a power law from ν_1 to ν_2. In the observer frame, every frequency gets Doppler shifted by $D_+ = \gamma(1+\beta\cos\theta) > 0$, where $\beta = v/c > 0$ (RL equation 4.12b). The factor of $1+\beta\cos\theta$ is the usual Doppler shift (train-whistle effect), while the factor of γ arises from time dilation. So the whole forward jet spectrum shifts to higher frequencies, between $D_+\nu_1$ and $D_+\nu_2$.

[1] I am indebted to Julia Kregenow for naming this problem.

What about the height of the spectrum? Use the Lorentz invariant I_ν/ν^3. Denote by $\tilde{I}_\nu$ the specific intensity of the jet seen in the observer's frame. Then by Lorentz invariance:

$$\frac{\tilde{I}_\nu(D_+\nu)}{(D_+\nu)^3} = \frac{I_\nu(\nu)}{\nu^3} \tag{1}$$

implying $\tilde{I}_\nu(D_+\nu) = D_+^3 I_\nu(\nu)$. So the forward jet spectrum gets "Doppler boosted" in intensity by a factor of D_+^3 (which can be $\gg 1$). To sum up: the forward jet spectrum is a power-law that extends from $D_+\nu_1$ to $D_+\nu_2$, whose amplitude at $D_+\nu_1$ equals $D_+^3 I_\nu(\nu_1)$, and whose amplitude at $D_+\nu_2$ equals $D_+^3 I_\nu(\nu_2)$. The spectral slope remains equal to α.

The answer for the counter-jet is the same, only now the Doppler factor equals $D_- = \gamma(1-\beta\cos\theta)$. Note that D_- can still be $\gg 1$, so that the emission from the counter-jet still gets Doppler boosted to higher frequency and higher intensity (just not quite as high as for the forward jet).

(b) By what (symbolic) factor is the forward jet brighter than the counter-jet in specific intensity at a fixed frequency?

Both jet and counter-jet spectra are power laws with the same slope. Therefore we can look at any single frequency and find the ratio of specific intensities. Let's choose $\nu = D_+\nu_1$. The forward jet's specific intensity at $\nu = D_+\nu_1$ equals $D_+^3 I_\nu(\nu_1)$. The counter-jet's specific intensity at $\nu = D_+\nu_1$ equals $D_-^3 I_\nu(\nu_1) \times [(D_+\nu_1)/(D_-\nu_1)]^\alpha$, where the correction factor arises because we have to scale from $\nu = D_-\nu_1$ to $\nu = D_+\nu_1$, and we know that the counter-jet spectrum scales as ν^α. Divide the forward jet specific intensity by the counter-jet specific intensity to find the desired ratio:

$$\left(\frac{D_+}{D_-}\right)^{3-\alpha} = \left(\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right)^{3-\alpha} \tag{2}$$

which does not depend explicitly on γ.

(c) The dynamic contrast in optical surface brightness between jet and counter-jet in M87 exceeds 450. If $\alpha = -1/2$, what are the constraints on γ, v, and θ?

Use the answer in part (b). Define $x = \beta\cos\theta$ and $y = 450$ to write

$$\left(\frac{1+x}{1-x}\right)^{7/2} > y \tag{3}$$

which can be solved for

$$1 > x = \beta \cos\theta > \frac{y^{2/7} - 1}{y^{2/7} + 1} = 0.7 \tag{4}$$

The maximum of β is one, so $\cos\theta > 0.7$ (the jet can't point too far away from the head-on direction). For a head-on jet, $\cos\theta = 1$ and $\beta > 0.7$. So $1 > \beta > 0.7$, with a corresponding constraint on $\gamma = 1/\sqrt{1-\beta^2} > 1.4$.

Problem 2. *In Space, No One Can Hear You Scream*

Take a look at Figure 3, Table 3, and Figure 4 of Carilli et al.'s paper on the radio galaxy Cygnus A.

In Figures 4a and 4b, we can distinguish a break in the spectral index at about $\nu \approx 10^4$ *MHz. The break is interpreted to represent the onset of synchrotron losses, as we studied in a previous problem. The corresponding electron energy spectrum will also be broken.*

In either regime, provided we don't look at frequencies too low (lower than $\nu \approx 10^{2.75}$ *MHz), the emission arises from optically thin material.*

In Figure 3, one arcsecond equals one kpc.

Part (d) is independent of the other parts.

(a) Estimate the minimum magnetic field strength in "Hot Spot A" using Burbidge's minimum energy argument and using Figure 3, Table 3, and/or Figure 4, or any other raw data from Carilli et al.'s paper. Express in Gauss.

You will have to decide whether to use the spectrum below the break, above the break, or both. We want a minimum estimate of the magnetic field.

Interpret the unit of "(beam)$^{-1}$*" (per beam) to mean* $[\pi(2.25\,\text{arcseconds})^2]^{-1}$*. This is the angular area of the radio interferometer's beam, or footprint on the sky. When Carilli et al. cite "4.5 arcseconds resolution," I interpret the 4.5 arcseconds to be equal to the FWHM (full-width-half-maximum) of their beam.*

Burbidge's minimum energy argument relies on a measurement of j_ν, the volume emissivity (erg/s/cm^3/Hz/sr). We can measured j_ν for optically thin media, and we can certainly measure it given the data from Carilli et al. The first question is: at what frequency should we measure j_ν?

The frequency of choice is the frequency at which the electrons carrying the bulk of the (kinetic) energy density are radiating. We stated in class that in many situations, the bulk of the energy density is carried by the lowest γ electrons, because they are overwhelmingly the most numerous. This is true whenever p (defined such that $dN/dE \propto E^p$) is less than -2. What is p in the hot spot? Synchrotron theory relates p to the spectral index, α (defined such that $F_\nu \propto \nu^\alpha$), via

$$\alpha = \frac{1+p}{2} \,. \tag{5}$$

Now we can measure α for the hot spot. In Table 3, the spectral slope between 1350 MHz and 4995 MHz—the portion of the spectrum at frequencies below the break—is $\alpha = -0.73$, which implies $p = -2.4 < -2$. Therefore yes, the bulk of the energy density ($\propto (dN/dE) \times dE \times E \propto E^{2+p}$) is carried by electrons having the lowest individual energies E (or γ) that radiate at the lowest frequencies ($\gamma^2 \omega_{cyc}$). From Figure 4c, the lowest frequency for which the power-law extends is about $\nu_1 = 10^{2.8}$ MHz. That is the frequency at which j_ν should be measured to obtain a lower bound on the total energy density. (Note that we need not worry about the spectrum at frequencies higher than the break because the spectrum there is still steeper than the spectrum at frequencies lower than the break.)

Let's go ahead and measure j_ν at ν_1. Now

$$j_\nu \sim \frac{L_\nu}{V} \tag{6}$$

where $V \sim (4/3)\pi r^3$ is the volume of the hot spot. Looking at Figure 3, I'd say the hot spot was a sphere about 5 arcseconds in radius, or $r = 5$ kpc. What is the luminosity density, L_ν, of this luminous sphere? (Note that density here refers to per Hertz, not per volume!) Define d to be the distance to the source and F_ν to be the flux density of the source. Then

$$L_\nu \sim 4\pi d^2 F_\nu \tag{7}$$

for an isotropic emitter. The flux density at ν_1 is $F_\nu \sim 10^2$ Jy/beam $\times\ \Delta\Omega$, where $\Delta\Omega$ is the solid angle subtended by the hot spot. Measured in beams, we estimate $\Delta\Omega \sim \pi(5\,\mathrm{arcsec})^2/\pi(2.25\,\mathrm{arcsec})^2 \sim 4$ beams. So $F_\nu \sim 400\,\mathrm{Jy}$. Since 1 arcsecond = 1 kpc, the distance to the source is $d = 2 \times 10^5$ kpc. Then putting it all together,

$$j_\nu \sim 8 \times 10^{-34}\,(\mathrm{cgs}) \tag{8}$$

This estimate is about 1 order of magnitude lower than curve 1 (labelled "Hot Spot A") of Figure 5a. I am not sure why the values are discrepant, but I suspect it has partly to do with the definition of "beam" (which Carilli et al. never define) and partly due to the modelled dimensions of the hot spot. It is possible that Carilli et al. are using a smaller volume for the hot spot. In any case, logic is more important than numbers for classwork, so we will proceed with our simpler and cleaner estimate.

Now we have to fill in the coefficients for Burbidge's minimum energy arguments as outlined in lecture. Start from first principles: the volume emissivity equals, to order-of-magnitude,

$$j_\nu \sim n_e \times P_{\rm single\,electron} \times \frac{1}{4\pi} \times \frac{1}{\nu} \tag{9}$$

where

$$P_{\rm single\,electron} = \frac{4}{3}\sigma_T c \gamma^2 \frac{B^2}{8\pi} \tag{10}$$

$$\nu = \frac{\gamma^2 \omega_{cyc}}{2\pi} (= \nu_1) \tag{11}$$

and n_e is the number density of electrons having γ. Burbidge's argument is based on minimization of total energy density

$$p_{tot} = p_e + p_{mag} \tag{12}$$

where the electron pressure p_e and the magnetic pressure p_{mag} are given by, respectively,

$$p_e \sim n_e \gamma m c^2 \tag{13}$$

$$p_{mag} = \frac{B^2}{8\pi} \tag{14}$$

Since Burbidge's argument is based on minimization of total energy density given a single measurement of j_ν, we must re-write j_ν in terms of p_{mag} and p_e. This was done in lecture without the coefficients. All the coefficients are here now before us, as given above. The same algebra that gave us the proportionality in lecture now gives us the order-of-magnitude expression

$$j_\nu \sim p_e p_{mag}^{3/4} \nu_1^{-1/2} C \tag{15}$$

$$C = \frac{2\sigma_T}{3\sqrt{2\pi e m c}(8\pi)^{1/4}} = 6 \times 10^{-13}\ ({\rm cgs}) \tag{16}$$

We performed the minimization in lecture and found that if we want to minimize p_{tot} given j_ν, then $p_{mag} \sim p_e \sim p_{tot}/2$, where

$$(min)\, p_{tot} \sim 2 \left(\frac{j_\nu \nu_1^{1/2}}{C} \right)^{4/7} \tag{17}$$

Putting it all together, $p_{mag} \sim 3.3 \times 10^{-10}\,\mathrm{dyne/\,cm^2}$, which means $B \sim 1 \times 10^{-4}$ G, or 100 μG. This is a factor of 3 lower than Carilli et al.'s declaration (without derivation) of 300 μG. The discrepancy can be traced mostly to the discrepancy in the estimate for j_ν; see above.

(b) Estimate the minimum total pressure—magnetic plus material (electron)—in hot spot A. Express in cgs units. Compare to the pressure of air around you.

From part (a), $p_{tot} \sim 7 \times 10^{-10}\,\mathrm{dyne/\,cm^2}$, compared to atmospheric pressure of $10^6\,\mathrm{dyne/\,cm^2}$. In space, even at the end of a relativistic jet, no one can hear you scream.

(c) Estimate the total energy (magnetic plus material) in hot spot A in ergs.

The total energy is $\sim p_{tot} V \sim 1 \times 10^{58}$ erg. And that's just for the hot spot, let alone the rest of the jet. Accretion onto a supermassive black hole at the jet origin is thought to give rise to such prodigious energies.

(d) Estimate an age of the hot spot given the observation of the spectral break. Compare to the dynamical time of the jet (the time it took the jet to grow to its present size).

The break occurs at $\nu \approx 10^4$ MHz. The synchrotron lifetime of electrons is

$$t_{\rm life} \sim \frac{\gamma m c^2}{(4/3)\sigma_T c \gamma^2 p_{mag}} \tag{18}$$

We use $\nu = \gamma^2 eB/2\pi mc \approx 10^4$ MHz and our estimate of B from part (a) to estimate $\gamma \sim 6 \times 10^3$. Then $t_{\rm life} \sim 5 \times 10^5$ yr.

The dynamical time is $t_{\rm dyn} \sim l/c$, where l is the length of the jet and we assume that it's moving at very nearly the speed of light.[2] Lengthwise, it's about 6 seconds of time in right ascension = 90 arcseconds of angle = 90 kpc = 3×10^5 light years. The true length might be a factor of few longer than this, because we only see the projection of the jet in the plane of the sky. In any case, a dynamical age of 3×10^5 yr is hearteningly comparable to our estimate of the age of the source based on the synchrotron break, t_{life}.

Problem 3. *Galactic Synchrotron Emission*

[2]Actually, as far as I can tell from the introduction of Carilli et al., we don't actually know the velocity of the jet. The velocity is INFERRED based on the synchrotron break, assuming equipartition field, and the requirement that $t_{\rm life} < t_{\rm dyn}$! This is an example of extreme faith in models.

The brightness temperature of synchrotron emission in the Galactic plane is measured to be

$$T_b = 250 \left(\frac{\nu}{480\mathrm{MHz}} \right)^{-2.8} \mathrm{K}\,, \tag{19}$$

valid for 8 GHz $> \nu >$ *480 MHz. This emission arises from cosmic ray electrons gyrating in the Galactic magnetic field of strength* $B \sim 3\mu G$*. Take the size of the emitting region to be* ∼*10 kpc.*

(Aside from being a nuisance foreground for CMB researchers, Galactic synchrotron emission is thought to probe the supernova rate in the Galaxy, since cosmic ray electrons and protons at the relevant energies are thought to be accelerated in supernova shock waves. In addition, a famous "FIR-radio correlation" is observed to exist for normal spiral galaxies; the radio traces electrons accelerated by supernovae, while the far-infrared (FIR) emission traces warm dust grains. Where there are supernovae, there is active star formation; where there is star formation, there are dust grains warmed by the ISRF. Or so goes the traditional interpretation of the FIR-radio correlation.)

(a) Provide an approximate expression for the differential energy spectrum of cosmic ray electrons. Express in units of particles $\mathrm{m^{-2}\,sr^{-1}\,s^{-1}\,GeV^{-1}}$*. Indicate the approximate range of energies (*$E_{\min}$ *and* $E_{\max}$*) for which your expression is valid.*

We solve this in three steps. The first step is estimate the number density of electrons, $n_e(\gamma_{\min})$ responsible for emission at $\nu = \nu_{\min} = 480$ MHz. The second step is to calculate the desired differential energy spectrum at electron energy $E_{\min}$ using $n_e(\gamma_{\min})$. The third step is to calculate the slope of the differential energy spectrum.

First $n_e(\gamma_{\min})$. By definition of the brightness temperature, and for optically thin media,

$$\frac{2kT_b}{\lambda^2} \sim j_\nu L \tag{20}$$

where $L \sim 10$ kpc and

$$j_\nu \sim \frac{(\frac{4}{3}\gamma^2\beta^2 U_B \sigma_T c) \times n_e(\gamma)}{\nu \times 4\pi} \tag{21}$$

The term in parentheses in the numerator is the total (bolometric) power radiated by a single electron, $n_e(\gamma)$ is the number density of electrons having Lorentz factor γ (in a logarithmically sized interval about γ), the ν in the denominator is thrown in because we are interested in j_ν (per Hertz), and the 4π is thrown in because j_ν is per steradian and we assuming isotropic emission from a tangled field.

Now $\nu \sim \gamma^2 \nu_{\rm cyc} \sim \gamma^2 eB/(2\pi m_e c)$. Also, for synchtrotron radiation, $\beta \approx 1$. We evaluate (21) at $\nu = \nu_{\rm min} = 480$ MHz, which corresponds to $\gamma = \gamma_{\rm min}$ and $T_b = 250$ K. Solve for $n_e(\gamma_{\rm min})$:

$$n_e(\gamma_{\rm min}) \sim \frac{2kT_b\nu_{\rm min}^2}{c^2} \times \frac{4\pi\nu_{\rm cyc}}{(\frac{4}{3}\frac{B^2}{8\pi}\sigma_T c) \times L} \tag{22}$$

We set $B = 3\mu$G, as given. After the arithmetic clears, we find $n_e(\gamma_{\rm min}) \sim 7 \times 10^{-12}\,{\rm cm}^{-3}$. (This is awfully small density. Most electrons in the universe are not of this high energy variety.)

Now the second step. Geometrical and dimensional considerations give

$$\frac{dN_e}{dE}(E_{\rm min}) \sim \frac{n_e(\gamma_{\rm min}) \times c}{4\pi \times \gamma_{\rm min} m_e c^2} \tag{23}$$

The left-hand side is the desired differential energy spectrum for the electrons (subscript "e"), evaluated at electron energy $E = E_{\rm min}$, in units of electrons per m^2 per s per sr per GeV. The c in the right-hand side numerator is thrown in to get a flux, because a number density times a speed equals a flux, and the speed of these synchrotron emitting electrons is the speed of light. The 4π is thrown in because this electron flux is isotropic (because we are assuming a tangled field, and an isotropic distribution of pitch angles), and because the desired dN_e/dE has units of per sr. The $\gamma_{\rm min} m_e c^2$ is thrown in because the desired dN_e/dE is a differential spectrum, i.e., it measures number of electrons per energy interval.

The only quantity we need to calculate in the above expression is $\gamma_{\rm min}$. That's easy: recall that $\nu = \nu_{\rm min} \sim \gamma_{\rm min}^2 \nu_{\rm cyc}$. Solve for $\gamma_{\rm min} \sim 8 \times 10^3$. Then (23) yields

$$\frac{dN_e}{dE}(E_{\rm min}) \sim 42\,{\rm m}^{-2}\,{\rm sr}^{-1}{\rm GeV}^{-1}\,{\rm s}^{-1} \tag{24}$$

where $E_{\rm min} = \gamma_{\rm min} m_e c^2 = 4$ GeV.

Finally, the third step is to calculate the slope. The measured flux density $F_{\rm nu} = 2kT_b/\lambda^2 \propto \nu_{-2.8}\nu^2 \propto \nu^{-0.8}$. By definition of the spectral index α defined in class, $\alpha = -0.8$. Now we also saw in class (and in the reading) that α is related to the electron energy spectral index by $\alpha = (1+p)/2$, where $dN_e/dE \propto E^p$. So $p = -2.6$ and

$$\frac{dN_e}{dE} \sim 42 \left(\frac{E}{4\,{\rm GeV}}\right)^{-2.6} {\rm m}^{-2}\,{\rm sr}^{-1}{\rm GeV}^{-1}\,{\rm s}^{-1} \tag{25}$$

This answer is good between $E_{\rm min} = 4$ GeV and $E_{\rm max} = 16$ GeV (the latter is calculated using $\nu_{\rm max} = 8$ GHz.

(b) Compare your answer to the differential energy spectrum of cosmic ray protons, as given in the reader on page 145. The units of the plot on page 145 were accidentally chopped off; they are the same as those specified in part (a). By what factor do cosmic ray protons outnumber cosmic ray electrons?

At $E = 4$ GeV, we compare our derived answer of 42 electrons (per everything) with the plot from the reader which gives about 300 protons (per everything). At $E = 16$ GeV, we get 1 electron (per everything) vs. 20 protons (per everything). So it seems the cosmic ray electrons are outnumbered by cosmic ray protons by a factor of 8–20, at fixed energy.

Textbooks like Shu and on-line sources quote a factor of 100 instead. So it seems our little calculation is off by a factor of 10. Perhaps the discrepancy can be partly resolved by restoring the various factors of order unity that we have ruthlessly discarded, and partly by setting the true size of the emitting region L to be larger than 10 kpc, say closer to 30 kpc.

(c) Estimate the gyro-radii of cosmic ray electrons and of cosmic ray protons at $E_{\rm min}$ and $E_{\rm max}$. Compare to the size of the (baryonic disk of the) Galaxy, 10 kpc. From this comparison you should be able to understand why cosmic ray astronomy is so difficult.

We know from class (and the reading) that the gyro-frequency $\omega_B = \omega_{\rm cyc}/\gamma$. The gyro-radius is just $v/\omega_B = c/\omega_B = \gamma mc^2/(eB)$. Notice this expression cares only about the total energy of the charged particle, γmc^2, regardless of whether it is an electron or a proton. So at $E_{\rm min} = \gamma_{\rm min} mc^2 = 4$ GeV, both the electron and proton gyro-radii are 4×10^{12} cm. At $E_{\rm max} = 16$ GeV, both the electron and proton gyro-radii are 1.6×10^{13} cm (about an AU). These gyro-radii are much smaller than object-to-object distances in the Galaxy (measured in parsecs to kiloparsecs), rendering cosmic ray astronomy difficult because the cosmic rays spewed out by some source (like a supernova or a magnetar) don't travel on straight lines to us (unlike photons).

Only when the gyro-radius exceeds the distance to the source is the curvature negligible. This becomes possible at, say, $E \sim 10^9$ GeV (near the "ankle" in the cosmic ray proton spectrum shown in the reader), when the gyro-radius is $\sim$1 kpc. So to locate astrophysical sources of cosmic rays, we must build detectors sensitive to those extreme energies.

(d) Why do we ignore the contribution from cosmic ray protons to the Galactic synchrotron emission at the above frequencies ν? Provide a quantitative answer.

Let's say

$$\left.\frac{dN_p}{dE}\right|_E = F \left.\frac{dN_e}{dE}\right|_E \tag{26}$$

where F is the factor by which protons outnumber electrons at fixed energy E. We derived $F \sim 10$, but more careful treatments seem to give $F \sim 100$, as noted in part (a). Let's say that F is a constant, and that both the protons and electrons obey $dN/dE \propto E^{-2.7}$ (splitting the difference between the reader which gives $p = -2.75$ for the protons and our answer in part (a) which gives $p = -2.6$ for the electrons). So the shapes of both distributions are the same; it's just the normalizations which are different.

If $F > 1$, why do we neglect the contribution of protons to the observed emission? Let's compute the contribution to j_ν from electrons, and compare to protons. From electrons,

$$j_{\nu,e} \sim \frac{n_e(\gamma_e)\sigma_{T,e}cU_B\gamma_e^2}{\nu} \tag{27}$$

So at fixed frequency ν,

$$\frac{j_{\nu,p}}{j_{\nu,e}} = \frac{n_p(\gamma_p)\gamma_p^2\sigma_{T,p}}{n_e(\gamma_e)\gamma_e^2\sigma_{T,e}} \tag{28}$$

Now $\sigma_T \propto m^{-2}$. So

$$\frac{j_{\nu,p}}{j_{\nu,e}} = \frac{n_p(\gamma_p)\gamma_p^2}{n_e(\gamma_e)\gamma_e^2}\left(\frac{m_e}{m_p}\right)^2 \tag{29}$$

At fixed ν, we have $\nu = \gamma_p^2\nu_{\rm cyc,p} = \gamma_e^2\nu_{\rm cyc,e}$, which implies $(\gamma_p/\gamma_e)^2 \propto m_p/m_e$. So

$$\frac{j_{\nu,p}}{j_{\nu,e}} = \frac{n_p(\gamma_p)}{n_e(\gamma_e)}\left(\frac{m_e}{m_p}\right)^1 \tag{30}$$

Finally, $n_p(\gamma_p) \propto (dN_p/dE)|_{E_p}E_p$, so $n_p(\gamma_p)/n_e(\gamma_e) = [(dN_p/dE)|_{E_p}/(dN_e/dE)|_{E_e}] \times (E_p/E_e) = F(E_p/E_e)^{-2.7} \times (E_p/E_e) = F(E_p/E_e)^{-1.7}$. Now $E_p/E_e = (\gamma_p m_p)/(\gamma_e m_e) = (m_p/m_e)^{3/2}$ from the line immediately preceding (30). So $n_p(\gamma_p)/n_e(\gamma_e) = F(m_e/m_p)^{2.55}$. Therefore

$$\frac{j_{\nu,p}}{j_{\nu,e}} = F\left(\frac{m_e}{m_p}\right)^{3.55} \sim 2 \times 10^{-10} \tag{31}$$

for $F = 100$. So the proton contribution at fixed ν is utterly negligible compared to the electron contribution (a repeated theme throughout this class).