# Problem Set 1 Solution

AY 7B

## Problem 1

(a) $R$ is stationary and $S$ is moving away at $v = \beta c_s$. Image two pulses: one emitted at time $t_1$ and when $S$ is at position $x_1$, and another emitted at time $t_2$ and when $S$ is at position $x_2$.

The first pulse is received by $R$ at time:

$$P_1 = t_1 + x_1/c_s, \tag{1}$$

and the second pulse is received at time:

$$P_2 = t_2 + x_2/c_s, \tag{2}$$

Now, the time between the two pulses as seen by $R$ is:

$$\Delta t' = P_2 - P_1 = (t_2 - t_1) + \frac{(x_2 - x_1)}{c_s}. \tag{3}$$

Now, $(t_2 - t_1)$ is the time interval between emission, $\tau_0 = \frac{1}{\nu_0}$. We can solve for $(x_2 - x_1)$ by noting that this is the distance $S$ covers during a time $\tau_0$. Therefore,

$$(x_2 - x_1) = \tau_0 \beta c_s = \frac{1}{\nu_0}\beta c_s. \tag{4}$$

Plugging this back to $\Delta t'$ gives us:

$$\Delta t' = \frac{1}{\nu_0}(1 + \beta). \tag{5}$$

We also know that $\Delta t' = \frac{1}{\nu'}$, where $\nu'$ is the frequency of the sound wave measured by the receiver. Using our previous equation for $\Delta t'$, we find the relationship between $\nu'$ and $\nu_0$ to be:

$$\frac{\nu'}{\nu_0} = \frac{1}{1 + \beta} \tag{6}$$

(b) Now, S is stationary and $R$ is moving (with respect to the medium). An important feature of waves that propagates through a medium (e.g. sound waves in air), is that the speed of propagation always refers to the speed in the medium (i.e. an observer at rest with the air will measure a speed of sound of $c_s$).

Now, when the observer is moving with respect to the medium, the speed of sound in the medium $c_s$ will be added with the relative speed between the medium and the observer (in this case $v_{net} = c_s - \beta c_s$). We can now form the same two-pulse picture as we did in part (a).

The first pulse is received by $R$ at time:

$$P_1 = \ t_1 + x_1/(c_s - \beta c_s), \tag{7}$$

and the second pulse is received at time:

$$P_2 = \ t_2 + x_2/(c_s - \beta c_s). \tag{8}$$

The time between the two pulses as seen by $R$ is:

$$\Delta t' = P_2 - P_1 = (t_2 - t_1) + \frac{(x_2 - x_1)}{c_s - \beta c_s}. \tag{9}$$

As per part (a), plug in $1/\nu_0$ for $(t_2 - t_1)$ and $\beta c_s/\nu_0$ for $(x_2 - x_1)$ to get:

$$\Delta t' = \frac{1}{\nu_0}\left(1 + \frac{\beta}{1-\beta}\right) = \frac{1}{\nu_0}\left(\frac{1}{1-\beta}\right). \tag{10}$$

Again, $\Delta t' = 1/\nu'$, so we can invert the equation to get:

$$\frac{\nu'}{\nu_0} = (1 - \beta) \tag{11}$$

## Problem 2: C&O 4.5

Throughout this problem, $\gamma^{-1} = \sqrt{1 - u^2/c^2} = 0.6$.

(a) $\Delta t_P = (60\ m)/(0.8c) = 0.250\ microseconds.$

(b) The rider sees a train at rest of length:

$$\text{Ł}_{rest} = \frac{L_{moving}}{\sqrt{1 - u^2/c^2}} = \frac{60\ m}{0.6} = 100\ m. \tag{12}$$

(c) The rider sees a moving platform of the length:

$$\text{Ł}_{moving} = L_{rest}\sqrt{1 - u^2/c^2} = (60\ m)(0.6) = 36\ m. \tag{13}$$

(d) $\Delta t_T = (100\ m)/(0.8c) = 0.417\ microseconds.$

(e) In the T frame, the train is 100 $m$ long and the platform is 36 $m$ long. The platform is moving with velocity $0.8c$ towards the train. The time $T$ measures for the platform to travel the extra 64 $m$ is $(64\ m)/(0.8c) = 0.267\ microseconds$

## Problem 3: C&O 4.6

Throughout this problem, $\gamma^{-1} = \sqrt{1 - u^2/c^2} = 0.6$.

(a) The two events are the starship leaving Earth and arriving at $\alpha$ Centauri. According to an observer on Earth, these events occur at different locations, so the time measured by a clock on Earth is:

$$(\Delta t)_{moving} = \frac{4ly}{0.8c} = 5\ yr. \tag{14}$$

(b) The starship pilot is at rest relative to the two events; they both occur just outside the door of the starship. According to the pilot, the trip takes:

$$(\Delta t)_{rest} = (\Delta t)_{moving}\sqrt{1 - u^2/c^2} = 3\ yr. \tag{15}$$

(c) Using $L_{rest} = 4\ ly$, the distance measured by the starship pilot may be found from':

$$L_{moving} = L_{rest}\sqrt{1 - u^2/c^2} = 2.4\ yr. \tag{16}$$

(d) According to Eq. (4.31) with $\Delta t_{rest} = 6\ months$ and $\theta = 0$, the time interval between receiving the signals aboard the starship is:

$$\Delta t_{obs} = \frac{\Delta t_{rest}(1 + u/c)}{\sqrt{1 - u^2/c^2}} = 18\ months. \tag{17}$$

(e) The same as part (d).

(f) From the relativistic Doppler shift,

$$\lambda_{obs} = \lambda_{rest}\sqrt{\frac{1 + v_r/c}{1 - v_r/c}} = 45\ cm. \tag{18}$$

## Problem 4: C&O 4.11, part (a)

This problem involves a lot of algebra, so bear with me: Measured in frame $S'$, the spacetime interval is:

$$(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2. \tag{19}$$

From the Lorentz transform equations:

$$(\Delta s')^2 = \left[\frac{c(\Delta t - u\Delta x/c^2)}{\sqrt{1 - u^2/c^2}}\right]^2 - \left[\frac{\Delta x - u\Delta t}{\sqrt{1 - u^2/c^2}}\right]^2 - (\Delta y)^2 - (\Delta z)^2. \tag{20}$$

Performing some algebra, we can reduce this equation to:

$$(\Delta s')^2 = \frac{(c^2 - u^2)(\Delta t)^2}{1 - u^2/c^2} + \frac{(u^2/c^2 - 1)(\Delta x)^2}{1 - u^2/c^2} - (\Delta y)^2 - (\Delta z)^2. \tag{21}$$

This can be further reduced to:

$$(\Delta s')^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta s)^2. \tag{22}$$

# Problem 5: C&O 4.13

Throughout this problem, $S$ is the Earth frame and $S'$ the starship A's frame. The velocity between the two frames is therefore $v_A = u = 0.8c$. From the frame of the Earth, the velocity of starship B is $v_B = -0.6c$. We can now use the Einstein velocity addition formula to find the velocity of starship B as measured from starship A:

$$v'_B = \frac{v_B - u}{1 - uv_B/c^2} = -0.946c. \tag{23}$$

The velocity of starship A as measured from starship B is just the opposite of this, $+0.946c$.

# Problem 6: C&O 17.6

(a) The time a photon takes to cross a frame of z-length $dz$ is $t = dz/c$. In this time, the frame has fallen a distance of:

$$\frac{1}{2}g(r)t^2, \tag{24}$$

towards the Sun, with $r$ being the distance between the frame and the Sun, and $g(r) = GM_{\odot}/r^2$ is the local gravitational acceleration. The distance perpendicular to the $z$-axis travelled by the frame during this time is:

$$\frac{1}{2}g(r)t^2 \; cos \; \alpha. \tag{25}$$

Now, refer to Fig. 17.10 and replace $l$ by $dz$, $\phi$ by $d\phi$, and $\frac{1}{2}gt^2$ by $\frac{1}{2}gt^2 \; cos \; \alpha$, resulting in:

$$\frac{\left[\frac{1}{2}g(r)t^2 \; cos \; \alpha\right]}{dz} = \frac{\left[\frac{dz}{2 \; cos(d\phi/2)}\right]}{\overline{OD}}. \tag{26}$$

Since $d\phi$ is small, we can set $cos(d\phi/2) \sim 1$ and $\overline{OD} \sim r_c$, the radius of curvature of the photon's path. Thus, in radians:

$$d\phi = \frac{dz}{r_c} = \frac{g(r)t^2 \; cos \; \alpha}{dz} = \frac{GM_{\odot}}{r^2}\left(\frac{dz}{c}\right)^2 \frac{cos \; \alpha}{dz}. \tag{27}$$

Defining $g_0 = GM_\odot/r_0^2$, where $r_0 = r\ cos\ \alpha$ is the distance of closest approach, we obtain:

$$d\phi = \frac{g_0\ (cos\ \alpha)^3}{c^2} dz. \tag{28}$$

(b) To find the total angular deflection, $\phi$, we integrate $d\phi$ from $\alpha = -\pi/2 + \pi/2$. Using $z = r_0\ tan\ \alpha$, we have $dx = r_0 (sec\ \alpha)^2 d\alpha$. Thus:

$$\phi = \int_{-\infty}^{\infty} g_0\ \frac{(cos\ \alpha)^3}{c^2} \mathrm{d}z = \frac{g_o r_o}{c^2} \int_{-\pi/2}^{\pi/2} cos\ \alpha\ \mathrm{d}\alpha = \frac{2 g_0 r_0}{c^2} = \frac{2GM_\odot}{r_0 c^2}. \tag{29}$$

Assuming the photon just grazes the Sun's surface, we set $r_0 = R_\odot$ and find:

$$\phi = \frac{2GM_\odot}{R_\odot c^2} = 4.24 \times 10^{-6}\ rad = 0.875''. \tag{30}$$

(c) The answer we got is half the correct value of 1.75". In general, the correct result for the angular deflection of a photon that passes within a distance $r_0$ of a spherical mass $M$ is:

$$\phi = \frac{4GM}{r_0 c^2}, \tag{31}$$

twice the previous result. Our derivation in parts (a) and (b) included only the effect of the curvature of space, and not the equal contribution of time running more slowly in the curved spacetime near the Sun. The photon spends more time near the Sun (as measured by a distant observer), and so suffers a larger angular deflection.

# Problem 7: C&O 17.7

(a) Elsewhere: $(\Delta s)^2 = (c\Delta t)^2 - (\Delta l)^2 < 0$.

(b) Future lightcone: $(\Delta s)^2 > 0$ and $t > 0$.

(c) Future lightcone: $(\Delta s)^2 > 0$ and $t > 0$.

(d) Elsewhere: $(\Delta s)^2 < 0$.

(e) Past lightcone: $(\Delta s)^2 > 0$ and $t < 0$.

(f) Future lightcone: $(\Delta s)^2 > 0$ and $t > 0$.

(g) Elsewhere: $(\Delta s)^2 < 0$.

(h) Elsewhere: $(\Delta s)^2 < 0$.