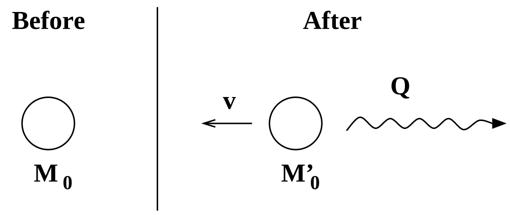
# Problem Set 2 Solutions

AY 7b

## Problem 1

An atom has rest mass $M_0$, and is initially at rest. It emits a photon and thus recoils in the opposite direction at speed $v$. After the emission of the photon, the atom has a new rest mass $M_0'$.




(a) Using the above picture as a guide, we use conservation of energy and momentum to give us two independent equations which we may solve for $\beta = v/c$ and the ratio $M_0'/M_0$.

Conservation of energy gives us:

$$M_0 c^2 = \gamma M_0' c^2 + Q \tag{1}$$

and conservation of momentum gives us:

$$\mathbf{p}_{net} = 0 = \gamma M_0' v - \frac{Q}{c} \tag{2}$$

or

$$M_0' = \frac{Q}{\gamma v c} \tag{3}$$

Using equation (2), we see that we can divide both sides by $M_0 c$ and rewrite it as:

$$\gamma \frac{M_0'}{M_0} \frac{v}{c} - \frac{Q}{M_0 c^2} = 0 \tag{4}$$

Defining $q = Q/(M_0 c^2)$, we get

$$q = \gamma \beta \frac{M_0'}{M_0} \tag{5}$$

Now plug equation (3) into equation (1), to get:

$$\begin{aligned}
M_0 c^2 &= \gamma c^2 \left( \frac{Q}{\gamma v c} \right) + Q \\
M_0 c^2 &= \frac{c}{v} Q + Q \\
1 &= \left( \frac{1}{\beta} + 1 \right) \frac{Q}{M_0 c^2} \\
1 &= \left( \frac{1}{\beta} + 1 \right) q \\
\frac{1}{q} &= \frac{1}{\beta} + 1
\end{aligned}$$

$$\boxed{\beta = \frac{q}{1-q}} \tag{6}$$

Now using equation (6), we see that

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\
&= \frac{1}{\sqrt{1-\frac{q^2}{(1-q)^2}}} \\
&= \frac{1-q}{\sqrt{(1-q)^2-q^2}}
\end{aligned}$$

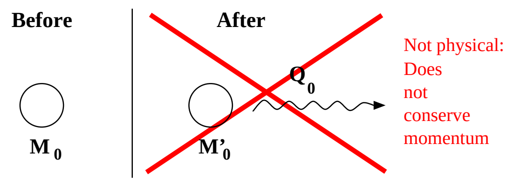
$$\gamma = \frac{1-q}{\sqrt{1-2q}} \tag{7}$$

Plugging equations (6) and (7) into equation (5), we see

$$q = \frac{M_0'}{M_0}\left(\frac{1-q}{\sqrt{1-2q}}\right)\frac{q}{1-q}$$

$$\boxed{\frac{M_0'}{M_0} = \sqrt{1-2q}} \tag{8}$$

(b) What if we were now able to ignore conservation of momentum, and the atom had not recoiled? It would instead emit a photon of energy $Q_0 = (M_0 - M_0')c^2$.




To find $Q/Q_0$, in terms of $q$, we write

$$Q_0 = M_0c^2\left(1-\frac{M_0'}{M_0}\right)$$

$$\frac{Q_0}{M_0c^2} = 1 - \frac{M_0'}{M_0}$$

Since $q = Q/(M_0c^2)$ and $M_0'/M_0 = \sqrt{1-2q}$, we have

$$\boxed{\frac{Q}{Q_0} = \frac{q}{1-\sqrt{1-2q}}} \tag{9}$$

## Problem 2: C&O 17.16

(a) To find the coordinate speed of light in the $\phi$ direction (for light in a Schwarzschild metric), we use the Schwarzschild metric (Eq. 17.22), which is

$$ds^2 = \left(cdt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2 \tag{10}$$

Light follows null spacetime intervals, so $ds = 0$. To see what the $\phi$ component of the speed of light, consider the special case of $dr = d\theta = 0$ and $\theta = 90°$, which simplifies the metric for this situation to

$$0 = \left(cdt\sqrt{1 - 2GM/rc^2}\right)^2 - (rd\phi)^2 \tag{11}$$

So we see that

$$\boxed{v_\phi = r\omega = r\frac{d\phi}{dt} = c\sqrt{1 - \frac{2GM}{rc^2}}} \tag{12}$$

(b) Because equation (17.26), $v = \sqrt{GM/r}$, does not depend on the mass of the particle orbiting the Schwarzschild black hole, we can directly use this equation in the limit that $m \to 0$, and use it for photons. Since this gives the particle speed for a purely circular orbit, $v_\phi$, we equate this with the result from part (a) to find a photon's circular orbit radius around the black hole:

$$\sqrt{\frac{GM}{r}} = c\sqrt{1 - \frac{2GM}{rc^2}}$$

$$\boxed{r = \frac{3GM}{c^2} = 1.5R_S} \tag{13}$$

using the Schwarzschild radius $R_S = 2GM/c^2$.

(c) Using the result from parts (a) and (b), we know that for the orbit at $r = 1.5R_S$ (this circular orbit) $v_\phi = c\sqrt{1 - 2GM/rc^2}$, so

$$v_\phi = r\frac{d\phi}{dt} = c\sqrt{1 - \frac{2GM}{c^2}\frac{c^2}{3GM}} = c/\sqrt{3}$$

The orbital period at $r = 1.5R_s$ is then

$$P = \int_0^P dt = \int_0^{2\pi} \frac{r}{c/\sqrt{3}}\, d\phi = \frac{(2\pi)3\sqrt{3}R_s}{2c} = \frac{3\pi\sqrt{3}R_s}{c}$$

For a $M = 10M_\odot$ black hole, $R_S = 2GM/c^2 = 29.6\,\text{km}$, so $\boxed{P = 1.61 \times 10^{-3}\,\text{s}}$.

(d) If a flashlight were beamed in the $\phi$ direction at $r = 1.5R_S$ near a Schwarzschild black hole, the photons would orbit around the black hole. We expect this because we already found that at $r = 1.5R_S$ photons travelling in the $\phi$ direction follow a circular orbit around the black hole.

## Problem 3: C&O 17.24

*Since this problem is numerical, there might be rounding differences. There should be no points taken off for small differences in the results.*

(a) Equation (7.7) gives the mass function as the right-hand side of

$$\frac{m_c^3}{(m_s + m_c)^2}\sin^3 i = \frac{P}{2\pi G}v_{s,r}^3\,,$$

where $v_{s,r} = 457\,\text{km s}^{-1}$ is the radial velocity of the star, $m_s$, $m_c$ are the masses of the star and compact object, respectively, and $i$ is the inclination angle. Since the orbital period is $P = 0.3226$ day, we find that

$$\frac{m_c^3}{(m_s + m_c)^2}\sin^3 i = 6.35 \times 10^{33}\,\text{g} = 3.17\,M_\odot$$

Looking at the left-hand side of the mass relation, we see that even if $i = 90°$, since $m_s > 0$, we know that the left-hand side must be less than $m_c$. Thus $\boxed{m_c > 3.17\,M_\odot}$.

(b) From the center of mass relation, $m_c r_c = m_s r_s$, and assuming both orbits are roughly circular, $v_c = 2\pi r_c/P$, $v_s = 2\pi r_s/P$, so $m_s/m_c = r_c/r_s = v_c/v_s = (v_{c,r}/\sin i) = (v_{s,r}/\sin i)$, so we have Eq (7.5):

$$\frac{m_s}{m_c} = \frac{v_{c,r}}{v_{s,r}} = 0.0941$$

Using the result from part (a), and assuming $i = 90°$ so $\sin i = 1$, we have

$$\frac{m_c}{(m_s/m_c + 1)^2} \sin^3 i = \frac{m_c}{(m_s/m_c + 1)^2} = 3.17\, M_\odot \tag{14}$$

$$m_c = (3.17\, M_\odot)(0.0941 + 1)^2$$

$$\boxed{m_c = 3.79\, M_\odot} \tag{15}$$

Since in reality $\sin^3 i \leq 1$, this means that $m_c \geq 3.79\, M_\odot$.

(c) Now supposing $i = 45°$, we use the result from equation (14) and find

$$m_c = \frac{(3.17\, M_\odot)(0.0941 + 1)^2}{\sin^3 i} = \frac{(3.17\, M_\odot)(0.0941 + 1)^2}{(0.7071)^3}$$

$$\boxed{m_c = 10.7\, M_\odot} \tag{16}$$

## Problem 4: C&O 18.2

For the equilateral triangle formed by $M_1$, $M_2$, and $L_4$ (or $L_5$), the side length of the triangle (the separation between $M_1$ and $M_2$) is $s_1 = s_2 = a$, as shown in Fig 18.1. Eq (18.4)

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2$$

and Eq (18.7) (Kepler III for the orbital period)

$$\omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{G(M_1 + M_2)}{a^3}$$

are combined to give

$$\Phi = -G\left(\frac{M_1 + M_2}{a}\right)\left(1 + \frac{r^2}{2a^2}\right) \tag{17}$$

Now Eq (18.3) gives that

$$r_1 + r_2 = a \qquad \text{and} \qquad M_1 r_1 = M_2 r_2$$

so

$$r_1 = a\left(\frac{M_2}{M_1 + M_2}\right)$$

$$r_2 = a\left(\frac{M_1}{M_1 + M_2}\right)$$

So the law of cosines worked out in Eqs (18.5) and (18.6) for $s_1$ and $s_2$ give

$$a^2 = a^2\left(\frac{M_2}{M_1 + M_2}\right)^2 + r^2 + 2a\left(\frac{M_2}{M_1 + M_2}\right) r\cos\theta$$

$$a^2 = a^2\left(\frac{M_1}{M_1 + M_2}\right)^2 + r^2 - 2a\left(\frac{M_1}{M_1 + M_2}\right) r\cos\theta$$

We use the one equation to get $\cos\theta$ in terms of the other variables, and then we solve the quadratic equation to get the distance $r$ between the center of mass and either $L_4$ or $L_5$

$$r = a\sqrt{1 - \frac{M_1 M_2}{(M_1 + M_2)^2}} \tag{18}$$

Plugging equation (18) into equation (17) gives

$$\boxed{\Phi(L_4) = \Phi(L_5) = -G\frac{M_1 + M_2}{2a}\left[3 - \frac{M_1 M_2}{(M_1 + M_2)^2}\right]} \tag{19}$$

Rewriting this in units of $G(M_1 + M_2)/a$, and plugging in the values from Figure 18.3, we get the same value as reported in Figure 18.3:

$$\boxed{\Phi(L_4) = \Phi(L_5) = -\frac{1}{2}\left[3 - \frac{M_1 M_2}{(M_1 + M_2)^2}\right] = -1.431} \tag{20}$$

# Problem 5: C&O 18.4

Eq (18.19) for the disk temperature, written with $x = R/r$, is

$$T = T_{disk}\, x^{3/4}(1 - \sqrt{x})^{1/4} \tag{21}$$

We want to find the maximum disk temperature, so we solve for $x$ where $dT/dx = 0$. So we find

$$\begin{aligned}\frac{dT}{dx} &= \frac{3}{4}x^{-1/4}(1 - \sqrt{x})^{1/4} - \frac{1}{4}(1 - \sqrt{x})^{-3/4}\frac{x^{3/4}}{2\sqrt{x}} \\ &= \frac{3}{4}x^{-1/4}(1 - \sqrt{x})^{1/4} - \frac{1}{4}(1 - \sqrt{x})^{-3/4}\frac{x^{1/4}}{2} = 0\end{aligned}$$

So we can solve to find

$$\begin{aligned}&3x^{-1/4}(1 - \sqrt{x}) - \frac{x^{1/4}}{2} = 0 \\ &3(1 - \sqrt{x}) - \frac{\sqrt{x}}{2} = 0 \\ &3 = \frac{7}{2}\sqrt{x} \\ &\boxed{x = \frac{36}{49}}\end{aligned}$$

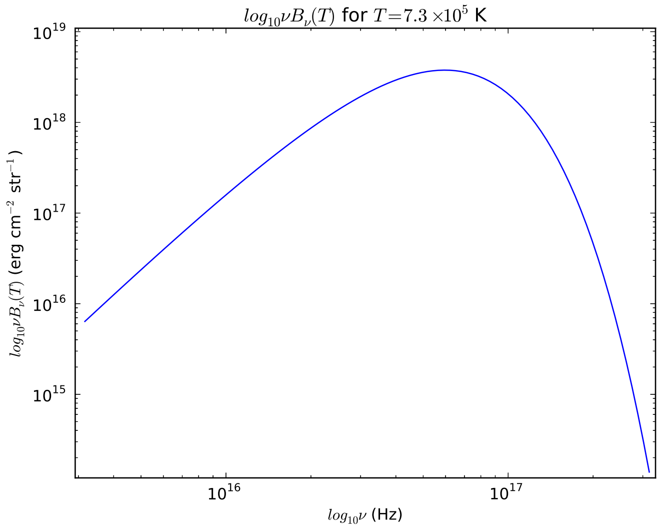
So $T$ is maximized when $r = (49/36)R$. Plugging this into equation (21) gives $\boxed{T_{max} = 0.488 T_{disk}.}$

# Problem 6: C&O 28.3

For $T = 7.3 \times 10^5$ K, we plot the Planck function (Eq 3.24), which is

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \tag{22}$$

We choose to plot $log_{10}\nu B_\nu(T)$ vs. $log_{10}\nu$ over the range $log_{10}\nu = [15.5, 17.5]$. In cgs units, $c = 3 \times 10^{10}$ cm/s, $k = 1.38 \times 10^{-16}$ erg/K, and $h = 6.62 \times 10^{-27}$ erg s.




Our simple Planck function for $T = 7.3 \times 10^5$ K is much more sharply peaked than the very broad spectrum of the quasar (3C 273) from Figure 28.14.

## Problem 7: C&O 28.11

(a) For material ejected from a quasar directly towards Earth (so moving entirely radially), the redshift of the quasar is $z_Q$ and the redshift of the ejecta is $z_{eq}$. To get the relative speed between the ejecta and the quasar, we start with the relativistic velocity transformation, Eq (4.40), which gives

$$\frac{v'_x}{c} = \frac{v_x/c - u/c}{1 - uv_x/c^2}$$

As in Figure 4.2, here we're assuming Earth is at the origin of frame $S$ (our observer's rest frame), and the quasar is at the origin of frame $S'$, which is moving with speed $u/c = v_Q/c = \beta_Q$ relative to Earth. From the Earth's frame, the ejecta has speed $v_x/c = v_{ej}/c = \beta_{ej}$, and from the quasar's frame, it has speed $v'_x/c = v'_{ej}/c = \beta'_{ej}$. (Since the material is ejected towards earth, $v'_x < 0$.)

We can then write the velocity transformation as

$$\beta'_{ej} = \frac{\beta_{ej} - \beta_Q}{1 - \beta_{ej}\beta_Q} \tag{23}$$

Redshift is defined as

$$z_Q + 1 = \sqrt{\frac{1+\beta_Q}{1-\beta_Q}} \qquad \text{and} \qquad z_{ej} + 1 = \sqrt{\frac{1+\beta_{ej}}{1-\beta_{ej}}}$$

so we find

$$\beta_Q = \frac{(z_Q+1)^2 - 1}{(z_Q+1)^2 + 1} \qquad \text{and} \qquad \beta_{ej} = \frac{(z_{ej}+1)^2 - 1}{(z_{ej}+1)^2 + 1} \tag{24}$$

Inserting the equations (24) into equation (23) gives

$$\beta'_{ej} = \frac{\frac{(z_{ej}+1)^2-1}{(z_{ej}+1)^2+1} - \frac{(z_Q+1)^2-1}{(z_Q+1)^2+1}}{1 - \left[\frac{(z_{ej}+1)^2-1}{(z_{ej}+1)^2+1}\right]\left[\frac{(z_Q+1)^2-1}{(z_Q+1)^2+1}\right]}$$

$$\beta'_{ej} = \frac{[(z_{ej}+1)^2 - 1][(z_Q+1)^2+1] - [(z_{ej}+1)^2+1][(z_Q+1)^2-1]}{[(z_{ej}+1)^2 + 1][(z_Q+1)^2+1] - [(z_{ej}+1)^2-1][(z_Q+1)^2-1]}$$

So

$$\boxed{-\beta'_{ej} = \frac{(z_Q + 1)^2 - (z_{ej} + 1)^2}{(z_Q + 1)^2 + (z_{ej} + 1)^2}} \tag{25}$$

Where $\beta'_{ej} = v'_x/c$ is the $x'$-component of the ejecta velocity relative to the quasar. Since our direction choices has $v'_x < 0$, the relative velocity is negative, as it is directed away from the quasar and towards earth (in the $-x'$ direction).

(b) If $z_Q + 1 = 1.158$ for quasar 3C 273, then with $\beta_{ej} = -0.9842$ for the radio knot approaching Earth, we have for the ejecta an observed redshift of

$$z_{ej} + 1 = \sqrt{\frac{1 + \beta_{ej}}{1 - \beta_{ej}}} = 0.08924$$

So the speed of the radio knot relative to the quasar is then

$$\boxed{-\beta'_{ej} = \frac{v}{c} = \frac{(z_Q + 1)^2 - (z_{ej} + 1)^2}{(z_Q + 1)^2 + (z_{ej} + 1)^2} = 0.9882} \tag{26}$$

Which means the Lorentz factor (from the quasar's perspective) of the ejecta is $\boxed{\gamma = 1/\sqrt{1 - v^2/c^2} = 6.53.}$