Problem Set 6 Solutions

AY 7b

Spring 2012

Problem 1: C&O 27.2

(a) In C&O Figure 15.14, we can roughly measure that the major axis of the ring of the remnant of SN 1987A is $a = 19$ mm, and the minor axis is $b = 14$ mm. So if we define θ to be the angle between the plane of the ring and the plane of the sky, then $\cos\theta = b/a = 0.7368$, so $\boxed{\theta = 42.5°.}$ [1]
(For this definition, if the ring wasn't tilted at all, and lay in the plane of the sky so we saw it as a perfect circle, the angle would be $\theta = 0$)

(b) If D is the linear diameter of the ring, then the light from the far side must travel an extra distance of $D\sin\theta$ to reach us on Earth. It took the light an extra $\Delta t = 340$ days to make the trip, so $D\sin\theta = c\Delta t$, or

$$\boxed{D = \frac{c\Delta t}{\sin\theta} = 1.3\times10^{18}\ \text{cm} = 0.422\ \text{pc}} \tag{1}$$

(c) Since the long axis of the light ring (as visible here on Earth) has a length D and subtends an angle $\alpha = 1.66'' = 8.05\times10^{-6}$ rad, the distance to SN 1987A is $\boxed{d = D/\alpha = 52.4\ \text{kpc.}}$

Problem 2: C&O 27.8

(a) From Kepler's third law (Eq. 2.37),

$$P^2 = \left(\frac{2\pi a}{v}\right)^2 = \frac{4\pi^2}{G(M_{\text{MW}} + M_{\text{gas}})}a^3. \tag{2}$$

We know that $a = 75$ kpc and $v = 244\ \text{km}\,\text{s}^{-1}$, and assuming that the mass of the clump of gas is negligible compared to the mass of the Milky Way (ie $M_{\text{gas}} \ll M_{\text{MW}}$), we can use Eq. (2) to solve for M_{MW}. Doing so gives

$$\boxed{M_{\text{MW}} = \frac{v^2 a}{G} = 2.06\times10^{42}\ \text{kg} = 1.04\times10^{12}\ \text{M}_\odot.} \tag{3}$$

From Table 24.1 we take that $L_B = 2.3\times10^{10}\ \text{L}_\odot$, so the mass-to-light ratio is $\boxed{M/L_B = 45\ \text{M}_\odot/\text{L}_\odot.}$

(b) Using conservation of energy and Eq. (2.14), we know that

$$\frac{1}{2}m\left(v_r^2 + v_t^2\right)_i - G\frac{Mm}{r_i} = \frac{1}{2}m\left(v_r^2 + v_t^2\right)_f - G\frac{Mm}{r_f}, \tag{4}$$

where M is the mass of the Milky Way and m is the mass of the clump of gas, and v_r and v_t are the radial and transverse components of the clump's velocity. The initial radial velocity is zero and the transverse velocity does not change, so we get that

[1] A more pleasing rounding gives exactly $\theta = 42°$!

$$M = \frac{v_r^2}{2G(1/r_f - 1/r_i)}. \tag{5}$$

We know that $r_i = 100$ kpc, $r_f = 50$ kpc, and the blob has a final radial velocity of $v_r = -220\ \mathrm{km\,s^{-1}}$, so we can estimate that $\boxed{M = M_{\mathrm{MW}} = 1.1 \times 10^{42}\ \mathrm{kg} = 5.6 \times 10^{11}\ \mathrm{M_\odot}.}$ Again using $L_B = 2.3 \times 10^{10}\ \mathrm{L_\odot}$, the mass-to-light ratio is $\boxed{M/L_B = 24\ \mathrm{M_\odot/L_\odot}.}$

Problem 3: C&O 27.11

For a flat rotation curve, Eq. (24.29) gives that $M_r = C_M r$, where C_M is a constant, and Eq. (24.50) shows that for a flat rotation curve, $\rho = C_\rho r^{-2}$, where C_ρ is a constant. (The mass & density described in these expressions include both dark matter and gas.) Let's write the temperature T in the form $T = C_T r^\alpha$, where C_T and α are constants. Eq. (27.17) can be rewritten as

$$C_M r = -\frac{kC_T r^{\alpha+1}}{\mu m_H G}\left[\frac{d\ln(C_\rho r^{-2})}{d\ln r} + \frac{d\ln(C_T r^\alpha)}{d\ln r}\right] \tag{6}$$

$$= -\frac{kC_T r^{\alpha+1}}{\mu m_H G}(-2+\alpha) \tag{7}$$

Since the left hand side of Eq. (7) depends on r^1, the right hand side must also depend on r^1, so it must be that $\boxed{\alpha = 0}$, so $\boxed{T = C_T r^\alpha = C_T r^0 = C_T}$ and the gas is indeed isothermal.

Problem 4: C&O 27.14

The radius of the Virgo cluster is $R = 1.5$ Mpc and its radial velocity dispersion is $\sigma_r = 666\ \mathrm{km\,s^{-1}}$. The virial mass (using Eq. 25.13) of the Virgo cluster is then

$$\boxed{M_{\mathrm{virial}} \approx \frac{5R\sigma_r^2}{G} = 1.5 \times 10^{45}\ \mathrm{kg} = 7.7 \times 10^{14}\ \mathrm{M_\odot}} \tag{8}$$

Problem 5: C&O 29.14

From the ideal gas law, $P = \rho kT/\mu m_H$. The thermal energy of an average hydrogen atom is $K = 3kT/2 = m_H v_{rms}^2/2$. If we substitute this expression for T into the ideal gas law, and we assume that we're dealing with pure neutral hydrogen (so $\mu = 1$), then we get that $\rho = P/(v_{rms}^2/3)$. When $v_{rms} = 600\ \mathrm{km\,s^{-1}}$, then we see that $\boxed{\rho = P/(v_{rms}^2/3) \gg P/c^2.}$

When $\rho = P/c^2$, it is necessary that

$$\frac{P}{\rho c^2} = \frac{kT}{\mu m_H c^2} = 1 \tag{9}$$

using the ideal gas law. The middle expression gives us that this corresponds to a temperature of $T = 1.1 \times 10^{13}$ K. (Yes, at this temperature the hydrogen would no longer be neutral so $\mu \neq 1$, but it's not that bad.) In an adiabatically expanding universe, $RT = T_0$, so

$$\boxed{R = \frac{T_0}{T} = \frac{2.725\ \mathrm{K}}{1.1 \times 10^{13}\ \mathrm{K}} = 2.5 \times 10^{-13},} \tag{10}$$

and this happens at a redshift of

$$\boxed{z = \frac{1}{R} - 1 = 4 \times 10^{12}.} \tag{11}$$

Problem 6: C&O 29.18

For a redshift of $z = 1.776$, the scale factor is $R = (1+z)^{-1} = 0.3602$ (using Eq. 29.4). Using Eq. 29.58, the temperature of the CMB was $\boxed{T = T_0/R = (2.726\ \text{K})/R = 7.57\ \text{K}}$, which agrees with the temperature of the intergalactic cloud at $T = 7.4 \pm 0.8$ K.