Ay 7B: Final Exam

Solutions

Spring 2012

Problem 1 (20 points)

(a) In the Sun's reference frame, the star is located at an angle θ above the x axis. So as photons have $v^2 = v_x^2 + v_y^2 = c^2$, and this photon is travelling towards the sun, we can simply write

$$\boxed{\begin{aligned} u_x &= -c\cos\theta \\ u_y &= -c\sin\theta \end{aligned}} \tag{1, 2}$$

(b) Now we want to transform the photon's velocity into the Earth's reference frame. The relative velocity between the reference frames is the Earth's orbital velocity (in the x-direction), $V_E = \beta_E c$, so the gamma factor is $\gamma = 1/\sqrt{1-\beta_E^2}$.

Using the relativistic velocity sum equations (and minding the directionality of u_x, u_y, and V_E), we get

$$u'_x = \frac{-c\cos\theta - \beta_E c}{1 + \beta_E c \frac{c\cos\theta}{c^2}} \tag{3}$$

$$u'_y = \frac{-c\sin\theta}{\gamma(1 + \beta_E c \frac{c\cos\theta)}{c^2}} \tag{4}$$

$$\boxed{\begin{aligned} u'_x &= \frac{-c(\beta_E + \cos\theta)}{1 + \beta_E\cos\theta} \\ u'_y &= \frac{-c\sin\theta}{\gamma(1 + \beta_E\cos\theta)} \end{aligned}} \tag{5, 6}$$

(c) In the Earth's frame, θ' is the observed angle of the star above the orbital plane. We know that the velocity components of the photon in the Earth's reference frame can be written as $u'_x = -c\cos\theta'$ and $u'_y = -c\sin\theta'$. So we can write

$$u'_x = -c\cos\theta' = \frac{-c(\beta_E + \cos\theta)}{1 + \beta_E\cos\theta} \tag{7}$$

So we find that

$$\boxed{\cos\theta' = \frac{\beta_E + \cos\theta}{1 + \beta_E\cos\theta}} \tag{8}$$

(d) When $\beta_E \ll 1$, our solution for (c) tells us that $\cos\theta \approx \cos\theta'$. So in the limit $\beta_E \ll 1$, we will find an expression for the difference $(\cos\theta' - \cos\theta)$.

Since $\beta_E \ll 1$, $1/(1 + \beta_E\cos\theta)$ is of the form $1/(1+\epsilon)$, which we Taylor expand as $1/(1+\epsilon) \approx 1 - \epsilon$. So

$$\cos\theta' \approx (\beta_E + \cos\theta)(1 - \beta_E\cos\theta) \tag{9}$$

$$= \beta_E - \beta_E^2\cos\theta + \cos\theta - \beta_E\cos^2\theta \tag{10}$$

Since β_E is very small, we can drop terms that are quadratic in β_E, so

$$\begin{aligned}\cos\theta' &= \beta_E - \cancel{\beta_E^2 \cos\theta} + \cos\theta - \beta_E \cos^2\theta &(11)\\ &= \beta_E + \cos\theta - \beta_E \cos^2\theta &(12)\end{aligned}$$

So we can write $(\cos\theta' - \cos\theta)$ as

$$\cos\theta' - \cos\theta = \beta_E(1 - \cos^2\theta) \tag{13}$$

Using the trig identity $\sin^2\theta + \cos^2\theta = 1$, we simplify this to

$$\boxed{\cos\theta' - \cos\theta = \beta_E \sin^2\theta} \tag{14}$$

Problem 2 (20 points)

(a) We can correlate the size of the accretion disk with the fluctuation of the continuum radiation because the continuum radiation emanates from the accretion disk surrounding the massive black hole:

$$r = c \times 1 \text{ hour} \sim 10^{14} \text{ cm} \sim 6 \text{ AU} . \tag{15}$$

(b) By similar reasonings, we can correlate the distance between the central black hole and the broad-line emission region with the fluctuation of the He II emission:

$$r = c \times 10 \text{ days} = 2.4 \times 10^{16} \text{ AU} . \tag{16}$$

(c) The thermal broadening is an application of the Doppler effect:

$$\frac{\Delta\lambda}{2\lambda} \sim 5.8 \times 10^8 \text{cm/s} \sim 0.02c . \tag{17}$$

For this question, we will also accept answers without the factor of two.

(d) Apply the usual equation relating the kinetic energy to temperature:

$$\frac{3}{2}kT = \frac{1}{2}mv^2 , \tag{18}$$

$$T \sim 5.4 \times 10^9 \text{K} . \tag{19}$$

Again, answers off by a factor of four due to dropping the 2 in the denominator of part (c) were given full points.

(e) Apply the Virial theorem:

$$\sigma^2 = \frac{GM_{BH}}{R_{BH}} , \tag{20}$$

$$M_{BH} \sim 1.6 \times 10^7 \ M_\odot . \tag{21}$$

Problem 3 (20 points)

(a) For circular motion:

$$\frac{GM}{R^2} = \Omega^2 R , \tag{22}$$

$$\Omega = \sqrt{\frac{GM}{R^3}} . \tag{23}$$

(b) Plug Ω straight to the epicyclic frequency $\mathcal{K}$ to get:

$$\mathcal{K}^2 = \frac{d}{dR}\left[\frac{GM}{R^2}\right] + 3\frac{GM}{R^3} \ . \tag{24}$$

$$\mathcal{K}^2 = \frac{GM}{R^3}(-2+3) = \frac{GM}{R^3} = \Omega^2 \ . \tag{25}$$

Therefore, $\mathcal{K}/\Omega = 1$.

(c) For rigid rotation, $v \propto R$, or $\Omega = A$ with A being some constant. Therefore:

$$\mathcal{K}^2 = \frac{d}{dR}\left[A^2 R\right] + 3A^2 = 4A^2 = 4\Omega^2 \ . \tag{26}$$

Therefore, $\mathcal{K}/\Omega = 2$.

(d) For constant rotation curve, $v \propto R^0$, or $\Omega = A/R$, giving:

$$\mathcal{K}^2 = \frac{d}{dR}\left[\frac{A^2}{R}\right] + 3\frac{A^2}{R^2} = \frac{A^2}{R^2}(-1+3) = 2\Omega^2 \ . \tag{27}$$

Therefore, $\mathcal{K}/\Omega = \sqrt{2}$.

Problem 4 (20 points)

(a) We find redshift as

$$z = \frac{\Delta\lambda}{\lambda_0} \tag{28}$$

or alternatively,

$$1+z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{\lambda_{\text{H}\alpha,\text{obs}}}{\lambda_{\text{H}\alpha,0}} = \frac{2.645\ \mu\text{m}}{6563\ \text{Å}} = \frac{2.645\times10^4\ \text{Å}}{6.563\times10^3\ \text{Å}} = 4 \tag{29}$$

So we find the redshift of the quasar as

$$\boxed{z = 3} \tag{30}$$

(b) Since we know that $1+z = \lambda_{\text{obs}}/\lambda_0$, the shortest observed wavelength will be that of Hδ. Thus we solve to find

$$\lambda_{\text{H}\delta,\text{obs}} = (1+z)\lambda_{\text{H}\delta,0} = 4(4102\ \text{Å}) = 1.64\times10^4\ \text{Å} \tag{31}$$

Since 10^4 Å $= 1\ \mu$m, we write this as

$$\boxed{\lambda_{\text{H}\delta,\text{obs}} = 1.64\ \mu\text{m}} \tag{32}$$

(c) To find the duration of the burst in the rest frame of the quasar, we use the fact that

$$\frac{\lambda_{\text{obs}}}{\lambda_0} = 1+z = \frac{\nu_0}{\nu_{\text{obs}}} = \frac{\Delta t_{\text{obs}}}{\Delta t_0} \tag{33}$$

Since we want to find $\Delta t_{\text{em}} = \Delta t_0$, we write

$$\Delta t_{\text{em}} = \Delta t_0 = \frac{\Delta t_{\text{obs}}}{1+z} = \frac{8\ \text{days}}{4} \tag{34}$$

So

$$\boxed{\Delta t_{\text{em}} = 2\ \text{days}} \tag{35}$$

(d) The present day proper distance is 6500 Mpc, and the proper distance today is equal to the comoving distance (since $R(t_0) = 1$). In general, the proper distance (d_{proper}) is related to the comoving distance (r) as

$$d_{\text{proper}} = Rr = \frac{r}{1+z} \tag{36}$$

where we have also used the relation $R = 1/(1+z)$ between the scale factor and the redshift.

So to find the proper distance to the quasar at the time the burst was emitted (at $z = 3$), we write

$$d_{\text{proper}}(z=3) = \frac{r}{1+z} = \frac{6500\ \text{Mpc}}{4} \tag{37}$$

$$\boxed{d_{\text{proper}}(z=3) = 1625\ \text{Mpc}} \tag{38}$$

Problem 5: Multiple Choice (20 points)

5.1 (c) [For a star moving *across* your line of sight,] $\nu_{\text{obs}} < \nu_\circ$: Time dilation increases the apparent period of the emitted light wave.

5.2 (a) [The total luminosity emitted by the disk depends on] the rate of mass transfer onto the star.

5.3 (d) [To observe molecular clouds, astronomers detect] radio emission from CO molecules.

5.4 (c) [The size of radio galaxy jets is about] 1 Mpc.

5.5 (c) [Must we use GR in calculating orbits of stars around the MW's central SMBH?] No, the stars are far outside the black hole's event horizon.

5.6 (a) [Why Sd galaxies have stronger emission lines than Sa galaxies:] Sd galaxies contain more massive stars that ionize gas.

5.7 (b) [Absolute B-magnitude of the MW?] -20.4

5.8 (c) [What happened to some globular clusters in Andromeda that we don't see?] They drifted into the disk of the galaxy, where they were tidally ripped apart.

5.9 (b) [Redshift z of a galaxy located 30 Mpc away:] 7×10^{-3}

5.10 (d) [Correct temporal order of cosmological events:] 4 - cosmological synthesis of helium nuclei from hydrogen nuclei, 2 - recombination of electrons and protons into hydrogen atoms, 1 - formation of the first galaxies, 3 - predominance of dark energy