Ay 7B: Midterm 1

Solutions

Spring 2012

Problem 1

(a) In the lab frame, $v = 0.99c$, so $\gamma = 7.09$. Also, converting m_p into GeV/c^2, we get $m_p = \frac{1.673\times10^{-27}\,\text{kg}}{1.783\times10^{-27}\,\text{kg}}\text{GeV} = 0.938\,\text{GeV/c}^2$.
Thus we use $p = \gamma m_p v = (7.09)(0.938\,\text{GeV/c}^2)(0.99c) \Rightarrow \boxed{p = 6.58\,\text{GeV/c}}$
We also use $E = \gamma m_p c^2 = (7.09)(0.938\,\text{GeV/c}^2)(c^2) \Rightarrow \boxed{E = 6.65\,\text{GeV}}$

(b) The frame moving along with the proton is the proton's *rest frame*, so we simply have that $p' = 0$ and $E' = m_p c^2$, so
$\boxed{p' = 0}$, $\boxed{E' = 0.938\,\text{GeV}}$

(c) In the CM frame, the velocity of the He nucleus is $v'_{He} = -V_{CM}$. Thus the momentum of the nucleus is

$$p'_{He} = \frac{-m_{He}V_{CM}}{\sqrt{1-\frac{V_{CM}^2}{c^2}}} \tag{1}$$

where $m_{He} = 4m_p$.

In the CM frame, the velocity of the proton is

$$v'_p = \frac{v_0 - V_{CM}}{1-\frac{V_{CM}v_0}{c^2}} \tag{2}$$

To find the new proton momentum, p'_p, we calculate

$$\gamma' = \frac{1}{\sqrt{1-\frac{(v'_p)^2}{c^2}}} \tag{3}$$

(Here, v_0 is the proton's velocity in the lab frame.) We will express γ' in terms of γ_0 (the Lorentz factor for the proton in the lab frame).

$$\frac{(v'_p)^2}{c^2} = \frac{\left(\frac{v_0}{c}-\frac{V_{CM}}{c}\right)^2}{\left(1-\frac{V_{CM}v_0}{c^2}\right)^2} \tag{4}$$

$$\begin{aligned} 1-\frac{(v'_p)^2}{c^2} &= \frac{\left(1-\frac{V_{CM}v_0}{c^2}\right)^2-\left(\frac{v_0}{c}-\frac{V_{CM}}{c}\right)^2}{\left(1-\frac{V_{CM}v_0}{c^2}\right)^2} &(5)\\ &= \frac{\left(1-\frac{v_0^2}{c^2}\right)\left(1-\frac{V_{CM}^2}{c^2}\right)}{\left(1-\frac{V_{CM}v_0}{c^2}\right)^2} &(6)\end{aligned}$$

So

$$\gamma' = \frac{\left(1 - \frac{V_{CM} v_0}{c^2}\right)}{\sqrt{1 - \frac{v_0^2}{c^2}}\sqrt{1 - \frac{V_{CM}^2}{c^2}}} \tag{7}$$

$$= \frac{\gamma_0 \left(1 - \frac{V_{CM} v_0}{c^2}\right)}{\sqrt{1 - \frac{V_{CM}^2}{c^2}}} \tag{8}$$

The new proton momentum is

$$p'_p = \gamma' m_p v'_p = \frac{\gamma_0 \left(1 - \frac{V_{CM} v_0}{c^2}\right)}{\sqrt{1 - \frac{V_{CM}^2}{c^2}}} m_p \frac{v_0 - V_{CM}}{1 - \frac{V_{CM} v_0}{c^2}} \tag{9}$$

So we see

$$p'_p = \frac{\gamma_0 m_p (v_0 - V_{CM})}{\sqrt{1 - \frac{V_{CM}^2}{c^2}}} = \tag{10}$$

$$-p'_{He} = \frac{4 m_p V_{CM}}{\sqrt{1 - \frac{V_{CM}^2}{c^2}}} \tag{11}$$

$$\Rightarrow \qquad \frac{v_0}{V_{CM}} - 1 = \frac{4}{\gamma_0} \tag{12}$$

Finally,

$$\frac{v_0}{V_{CM}} = 1 + \frac{4}{\gamma_0} = 1.564 \tag{13}$$

$$\boxed{V_{CM} = \frac{v_0}{1.564} = 0.633\, c} \tag{14}$$

(c) **(Alternate solution:)** For this part, let the unprimed frame be the lab frame (S), and let the primed frame be the CM frame (S').

We know that the 4-momentum, $\mathbf{p} = (E, \vec{p}c)$, lets us express an invariant quantity, the rest mass, for a system, since

$$\mathbf{p} \cdot \mathbf{p} = E^2 - p^2 c^2 = m^2 c^4 \equiv constant. \tag{15}$$

In the lab frame,

$$\mathbf{p}_{net} = (E_p + E_{He}, p_p c + p_{He} c) = (\gamma m_p c^2 + m_{He} c^2, p_p c) = ((\gamma + 4) m_p c^2, p_p c). \tag{16}$$

So the invariant for the system (proton and helium nucleus) between the lab frame and the CM frame is

$$\mathbf{p}_{net} \cdot \mathbf{p}_{net} = (\gamma + 4)^2 m_p^2 - (\gamma^2 - 1) m_p^2 c^4 = (8\gamma + 17) m_p^2 c^4 = m_{inv}^2 c^4, \tag{17}$$

since $m_p^2 c^4 = \gamma^2 m_p^2 c^4 - p_p^2 c^2$ implies that $p_p^2 c^2 = (\gamma^2 - 1) m_p^2 c^4$.

Now consider the CM frame. In this frame, the net momentum is $p'_{CM,net} = 0$, by construction. Thus

$$m_{inv}^2 c^4 = E'^2_{CM,net} - p'^2_{CM,net} c^2 = E'^2_{CM,net}, \tag{18}$$

and the net 4-momentum is just

$$\mathbf{p}'_{CM,net} = (m_{inv} c^2, 0). \tag{19}$$

But think again of how we've just expressed our 2-particle system in the CM frame: we wrote it in terms of the total invariant mass, which is conserved between the CM and lab frames. What if we backtrack, and consider the motion of the total invariant mass (as a single quantity m_{inv}) in the lab frame? Since the net invariant mass is at rest in the CM frame, and V_{CM} is the velocity between the lab and CM frames, we know that we can write (using $\gamma_{CM} = 1/\sqrt{1 - V_{CM}^2/c^2}$) that

$$\mathbf{p}_{net} = (\gamma_{CM} m_{inv} c^2, \gamma_{CM} m_{inv} V_{CM} c). \tag{20}$$

But because this is in the same frame as our original expression for $\mathbf{p}_{net}$, we know that the total energy and total momentum are the same, so

$$(\gamma + 4) m_p c^2 = \gamma_{CM} m_{inv} c^2 \tag{21}$$

and

$$p_p c = \gamma_{CM} m_{inv} V_{CM} c. \tag{22}$$

Now, it is most convenient to use the fact that $pc/E = v/c$, so

$$\frac{p_{net} c}{E_{net}} = \frac{p_{CM,net} c}{E_{CM,net}} = \frac{\gamma_{CM} m_{inv} V_{CM} c}{\gamma_{CM} m_{inv} c^2} = \frac{V_{CM}}{c} \tag{23}$$

$$\frac{\gamma m_p v c}{(\gamma + 4) m_p c^2} = \frac{V_{CM}}{c} \tag{24}$$

and we get

$$\boxed{V_{CM} = \frac{\gamma v}{(\gamma + 4)} = 0.63\, c = 1.89 \times 10^8 \text{ m/s}} \tag{25}$$

Problem 2

(a) The mass of a ring of differential thickness dR is:

$$dM = 2\pi R \Sigma(R) dR. \tag{26}$$

Now, it takes a time dt for this mass to drift inward a differential length dR, therefore:

$$\frac{dM}{dt} = 2\pi R \Sigma(R) \frac{dR}{dt}. \tag{27}$$

$\frac{dM}{dt}$ is exactly $\dot{M}$, and $\frac{dR}{dt}$ is exactly the drift velocity, $u(R)$. Therefore, rewriting:

$$\boxed{\dot{M} = 2\pi R \Sigma(R) u(R).} \tag{28}$$

(b) Consider a ring of some radial thickness l. If $\Sigma(R)$ is not a function of time, the interior mass of this ring must also be constant. The only way for this to hold is for the mass entering the ring's outer edge (from radius $r + l$) to be equal to the mass exiting the ring's inner edge (at radius r). This means $\dot{M}$ on the ring's outer edge must be equal to the $\dot{M}$ of the ring's inner edge.

(c) Use the fact that $\dot{M}$ is constant to get:

$$u \propto \frac{1}{R\Sigma}, \tag{29}$$

but Σ is proportional to $R^{-3/2}$, so:

$$u \propto \frac{1}{R \times R^{-3/2}} \propto R^{1/2}. \tag{30}$$

Using this proportionality, we can calculate:

$$\boxed{\frac{u_J}{u_E} = (5.2)^{1/2} = 2.3.} \tag{31}$$

Problem 3

(a) If the redshift is cased by the star's recession velocity, the redshift is due to the (relativistic) Doppler Effect – the observed redshift is entirely due to the relative motion of the emitter (the star) with respect you you, as you sit "at rest".
Assuming the star is moving away from us radially, the relativistic Doppler shift is given by

$$\nu_{obs} = \nu_{rest}\sqrt{\frac{1-v/c}{1+v/c}} = \nu_{rest}\sqrt{\frac{1-\beta}{1+\beta}} \tag{32}$$

So the redshift is given by

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \tag{33}$$

We solve to find

$$(1+z)^2 = \frac{1+\beta}{1-\beta} \tag{34}$$

$$(1-\beta)(1+z)^2 = (1+\beta) \tag{35}$$

$$(1+z)^2 - 1 = \beta(1+(1+z)^2) \tag{36}$$

$$\beta = \frac{(1+z)^2-1}{(1+z)^2+1} \tag{37}$$

So we find that

$$\boxed{v = c\left(\frac{(1+3)^2-1}{(1+3)^2+1}\right) = \frac{15}{17}c = \left(\frac{15}{17}\right)3\times10^{10}\text{ cm/s} = 2.6\times10^{10}\text{cm/s}} \tag{38}$$

(b) If the redshift of the spectral line is caused because the star is a neutron star, then the redshift is the result of gravitational redshift (where photons lose energy because they must climb out of a potential well).
For gravitational redshift caused by a central mass M, we have

$$z = \frac{\nu_0}{\nu_\infty} - 1 = \left(1 - \frac{2GM}{r_0c^2}\right)^{-1/2} - 1 \tag{39}$$

Thus we have

$$\frac{2GM}{r_0c^2} = 1-(1+z)^{-2} \tag{40}$$

$$r_0 = \frac{2GM}{c^2\,(1-(1+z)^{-2})} \tag{41}$$

To get the maximum radius, we see that we want to maximize M, so we will use $z=3$ and $M = 3\,M_\odot$ to get

$$\boxed{R = r_0 = \frac{2(6.67\times10^{-8}\text{cm}^3/\text{g}\,\text{s}^2)(3)(2\times10^{33}\text{g})}{(3\times10^{10}\text{cm/s})^2\,(1-(1+3)^{-2})} = 9.49\times10^5\text{cm} = 9.49\,\text{km}} \tag{42}$$

(c) This list is *not* exhaustive. (All responses exam responses will be considered carefully: there can be more possible measurements that could be made!)

(i) Measure line width – if there is Doppler broadening of line, it is likely a neutron star, since they tend to spin very rapidly.

(ii) Zeeman splitting of lines, as NS tend to have strong B fields compared to normal stars.

(iii) Time variability – NS are smaller, so have a smaller lower limit on the time variability they can exibit than a normal size star.

(iv) Presence of an accretion disk – an accretion disk would (a) probably not surround a normal star and (b) would have a small inner edge (so might play into the aforementioned time variability), and would emit greater relative power in X-rays and γ-rays around a NS. So observing the object in X-rays or γ-rays and seeing if the object is bright would suggest a NS with an accretion disk.

(v) Take a full spectrum and look at composition – normal stars have lots of H, He, and other lighter elements; the surface of NS is entirely Fe.