Ay 7B: Midterm 2

Solutions

Spring 2012

Problem 1 (20 points)

(a) Recall the absolute magnitude equation:

$$M_v = -2.5\log(F(10\ pc)/F_{ref}), \tag{1}$$

where $F(10\ pc)$ is the flux of the object at 10 parsecs and F_{ref} is the reference flux. How many $M_o = -2$ objects can we have before their combined magnitude reach $M_{tot} = -10$? First, let us write down the absolute magnitude equation for entire collection:

$$M_{tot} = -10 = -2.5\log(NF_o/F_{ref}), \tag{2}$$

where N is the number of objects and F_o is the flux of a single object at 10 parsecs. To get F_o/F_{ref}, we use the absolute magnitude equation for a single object:

$$M_o = -2 = -2.5\log(F_o/F_{ref}), \tag{3}$$

$$\frac{F_o}{F_{ref}} = 10^{0.8}. \tag{4}$$

Plugging this to the equation for M_{tot} gives us:

$$4 = 0.8 + \log(N), \tag{5}$$

$$N = 10^{3.2}$$
$$\boxed{N = 1585} \tag{6}$$

(b) Assuming no dust, the magnitude-distance relation is

$$m_v = M_v + 5\log_{10}(d) - 5 \tag{7}$$

For our cluster, $M_v = -10$, so we rewrite our magnitude-distance relation to give the distance d.

$$m_v = 5\log_{10}(d) - 15$$
$$d = 10^{(m_v+15)/5}\ \text{pc} \tag{8}$$

Telescope A, with the limiting magnitude of $m_v = +25$ successfully observed the globular cluster, meaning that $m_{GC} < m_{v_A} = +25$, so the distance to the globular cluster must be $d_{GC} < d\left(m_{v_A}\right) = 10^{(25+15)/5}\ \text{pc} = 10^8\ \text{pc}$.

Telescope B, with the limiting magnitude of $m_v = +20$ failed to observe the globular cluster, meaning that $m_{GC} > m_{v_B} = +20$, so the distance to the globular cluster must be $d_{GC} > d\left(m_{v_B}\right) = 10^{(20+15)/5}\ \text{pc} = 10^7\ \text{pc}$.

So the limits on the distance to the galaxy are

$$\boxed{10^7\ \text{pc} < d < 10^8\ \text{pc}} \tag{9}$$

Problem 2 (50 points)

(a) $\boxed{R_{max} = D\theta_{max} = 2.394 \times 10^{13} \text{ m} = 7.758 \times 10^{-4} \text{ pc.}}$

(b) The average number density is:

$$\boxed{n_* = \frac{M}{M_* \times Volume} = \frac{3M}{4M_* \pi R_{max}^3} = 6.52 \times 10^{-34} \text{ m}^{-3} = 1.916 \times 10^{16} \text{ pc}^{-3}.} \tag{10}$$

(c) We can use the virial theorem to get the answer:

$$\frac{GM}{R_{max}} = \frac{2}{2} v_*^2, \tag{11}$$

solving for v_*:

$$\boxed{v_* = \sqrt{\frac{GM}{R_{max}}} = 4.089 \times 10^6 \text{ m/s.}} \tag{12}$$

(d) The mean free path, l, is given by:

$$l = (n_* \sigma_*)^{-1}, \tag{13}$$

Where σ_* is the cross section of a brown dwarf interacting with another brown dwarf:

$$\sigma_* = \pi (r_{int})^2 = \pi (2R_*)^2 = 4\pi R_*^2. \tag{14}$$

Notice that r_{int}, the interaction radius is $2R_*$ for spherical brown dwarfs each with radius R_*. An easy way to think about this is to note that the center of a brown dwarf needs only to be as close as $2R_*$ away from the center of another brown dwarf for the two objects to interact.

The collision time, t_{coll}, is then given by:

$$t_{coll} = \frac{l}{v_*} = \frac{1}{n_* \sigma_* v_*} = \frac{1}{n_* 4\pi R_*^2} \sqrt{\frac{R_{max}}{GM}}$$

$$\boxed{t_{coll} = 7.516 \times 10^9 \text{ sec} = 238 \text{ years.}} \tag{15}$$

(e) The cluster is not a plausible alternative to the supermassive black hole. The collision time is far too fast! What happens when two brown dwarfs collides? Well, the resulting merged object can have enough mass to sustain hydrogen fusion: they become stars. If this theory is true, then we can detect these stars from Earth! Of course, we detect no such clump of stars at the center of our galaxy. Alternatively, these stars would keep merging and evolving, finally becoming supermassive black holes!

Problem 3 (30 points)

(a) For a density distribution $\rho(r)$, the enclosed mass inside a radius r is

$$dM(r) = \rho(r) dV = \rho(r)(4\pi r^2 dr) \tag{16}$$

Integrate to find

$$M(r) = \int_0^r dM(r') = \int_0^r 4\pi r'^2 \rho(r') dr' \tag{17}$$

So now solve for $M(r)$ in the three radii ranges. For $r < a$, $\rho(r) = \rho_0$, so

$$M(r) = \int_0^r 4\pi r'^2 \rho_0 dr' = \frac{4\pi}{3} r^3 \rho_0 \tag{18}$$

For $a < r < b$, $\rho(r) = \rho_0(a/r)$, so

$$M(r) = \int_0^a 4\pi r'^2 \rho_0 dr' + \int_a^r 4\pi r'^2 \rho_0 \left(\frac{a}{r'}\right) dr' \tag{19}$$
$$= \frac{4\pi}{3} a^3 \rho_0 + \int_a^r 4\pi \rho_0 a r' dr'$$
$$= \frac{4\pi}{3} a^3 \rho_0 + 2\pi \rho_0 a (r^2 - a^2) \tag{20}$$

Finally, for $r > b$, $\rho(r) = 0$, so

$$M(r) = \int_0^a 4\pi r'^2 \rho_0 dr' + \int_a^b 4\pi r'^2 \rho_0 \left(\frac{a}{r'}\right) dr' + \cancel{\int_b^r 4\pi r'^2 (0) dr'} \tag{21}$$
$$= \frac{4\pi}{3} a^3 \rho_0 + 2\pi \rho_0 a (b^2 - a^2) \tag{22}$$

So the overall mass enclosed is given as

$$\boxed{M(r) = \begin{cases} \frac{4\pi}{3} \rho_0 r^3 & r < a \\ \frac{4\pi}{3} \rho_0 a^3 + 2\pi \rho_0 a (r^2 - a^2) & a < r < b \\ \frac{4\pi}{3} \rho_0 a^3 + 2\pi \rho_0 a (b^2 - a^2) & r > b \end{cases}} \tag{23}$$

(b) Because we have a disk galaxy, we want to know the rotation curve defined by the circular velocity $\Theta(r)$. If we assume that $M(r)$ is much greater than the luminous matter in the galaxy at every r, then we can neglect the fact that the luminous matter is distributed in a disk, and just assume that our mass follows a purely spherical profile.

For a spherically symmetric mass distribution, the rotation curve is found by balancing the centripetal acceleration against the gravitational acceleration of the enclosed mass:

$$\frac{\Theta(r)^2}{r} = \frac{GM(r)}{r^2} \tag{24}$$

so we solve for $\Theta(r)$:

$$\Theta(r) = \sqrt{\frac{GM(r)}{r}} \tag{25}$$

and then we use the mass enclosed given in Eq (23) to find that, for $r < a$, we have

$$\Theta(r) = \sqrt{\frac{4\pi G \rho_0 r^3}{3r}} = \sqrt{\frac{4\pi G \rho_0}{3}}\, r \tag{26}$$

For $a < r < b$, we have

$$\Theta(r) = \sqrt{\frac{4\pi G \rho_0 a^3}{3r} - \frac{2\pi G \rho_0 a^3}{r} + \frac{2\pi G \rho_0 a r^2}{r}} = \sqrt{2\pi G \rho_0 a r - \frac{2\pi G \rho_0 a^3}{3r}} \tag{27}$$

For $r > b$, we have

$$\Theta(r) = \sqrt{\frac{4\pi G \rho_0 a^3}{3r} + \frac{2\pi G \rho_0 a (b^2 - a^2)}{r}} = \sqrt{\frac{4\pi}{3} G \rho_0 a^3 + 2\pi G \rho_0 a (b^2 - a^2)}\, \frac{1}{\sqrt{r}} \tag{28}$$

Together this describes the full rotation curve $\Theta(r)$:

$$\boxed{\Theta(r) = \begin{cases} \sqrt{\frac{4\pi G\rho_0}{3}}\ r & r < a \\ \sqrt{2\pi G\rho_0 ar - \frac{2\pi G\rho_0 a^3}{3r}} & a < r < b \\ \sqrt{\frac{4\pi}{3}G\rho_0 a^3 + 2\pi G\rho_0 a(b^2 - a^2)}\ \frac{1}{\sqrt{r}} & r > b \end{cases}} \tag{29}$$

(c) We find $v_{esc}(r)$ by balancing the kinetic energy necessary to escape (and assuming that this motion is entirely radial) with the potential energy at the same radius.

$$\begin{aligned} KE + U &= 0 \\ \frac{1}{2}mv_{esc}^2 &= \frac{GmM(r)}{r} \\ v_{esc}(r) &= \sqrt{\frac{2GM(r)}{r}} \end{aligned} \tag{30}$$

So to find $v_{esc}(b)$, we use our result from part (b):

$$\begin{aligned} v_{esc}(b) &= \sqrt{\frac{2G}{b}\left(\frac{4\pi}{3}\rho_0 a^3 + 2\pi\rho_0 a(b^2 - a^2)\right)} \\ &= \sqrt{\frac{2G}{b}\left(2\pi\rho_0 ab^2 - \frac{2\pi}{3}\rho_0 a^3\right)} \\ \Aboxed{v_{esc}(b) &= \sqrt{\frac{4\pi G\rho_0}{3}\ \frac{a\left(3b^2 - a^2\right)}{b}}} \end{aligned} \tag{31}$$