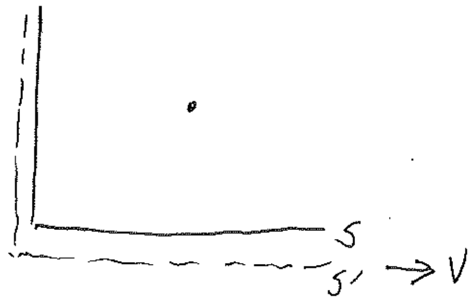
First
* Twin paradox
* inertial ref. frames

L2 – Special Relativity II

Velocity Transformation

In Newtonian physics, $V_x' = V_x \pm V$. This can no longer be true, because c is the ultimate speed.



Consider events (in S + S') that have two slightly different coordinates. These are:
x and $x+\Delta x$, y and $y+\Delta y$, z and $z+\Delta z$, t + $t+\Delta t$ in S, and $x' + \Delta x'$ etc in S'.

If these two events represent flashes on a material object, then its velocity, as measured in S, is $v_x = \frac{\Delta x}{\Delta t}$ etc, and, as measured in S', is $v_x' = \Delta x'/\Delta t'$.

To derive the transformation: $x' = \gamma(x - \beta c t)$ $\quad t' = \gamma(t - \frac{\beta x}{c})$
$y' = y \quad z' = z$

Differentiating:

$$\Delta x' = \gamma(\Delta x - \beta c \Delta t) = \gamma \Delta x - \gamma\beta c \Delta t \qquad \Delta y' = \Delta y$$
$$\Delta t' = \gamma(\Delta t - \frac{\beta}{c}\Delta x) = \gamma \Delta t - \frac{\gamma\beta}{c}\Delta x \qquad \Delta z' = \Delta z$$

$$v_x' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma\Delta x - \gamma\beta c \Delta t}{\gamma \Delta t - \frac{\gamma\beta}{c}\Delta x} \text{ dividing by } \gamma\Delta t \longrightarrow \frac{\Delta x/\Delta t - \beta c}{1 - \frac{\beta}{c}\frac{\Delta x}{\Delta t}}$$

$$\boxed{V_x' = \frac{V_x - V}{1 - \frac{V}{c^2}V_x}}$$

$$V_y' = \frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma\Delta t - \frac{\gamma\beta}{c}\Delta x} \text{ dividing by } \Delta t \longrightarrow \frac{\Delta y/\Delta t}{\gamma - \frac{\gamma\beta}{c}\frac{\Delta x}{\Delta t}}$$

$$\boxed{V_y' = \frac{V_y}{\gamma\left(1 - \frac{V}{c^2}V_x\right)}} \text{ and } \boxed{V_z' = \frac{V_z}{\gamma\left(1 - \frac{V}{c^2}V_x\right)}}$$

The inverse Xformations are:
$$\boxed{V_x = \frac{V_x' + V}{1 + \frac{V}{c^2}V_x'} \quad V_y = \frac{V_y'}{\gamma\left(1 + \frac{V}{c^2}V_x'\right)} \quad V_z = \frac{V_z'}{\gamma\left(1 + \frac{V}{c^2}V_x'\right)}}$$

Example: A Photon *

Suppose the particle has $V_x' = c$, $V_y' = 0$, $V_z' = 0$. What is V_x, etc?

$$V_x = \frac{V_x' + V}{1 + \frac{V}{c^2} V_x'} = \frac{c + V}{1 + \frac{V}{c^2} c} = \frac{c + V}{1 + \frac{V}{c}} = c \checkmark \qquad V_y = V_z = 0$$

Doppler Shift: Acoustic

This is a familiar situation: The fall in pitch of a car's siren as it passes by. Quantitatively, the fall depends on whether (a) the source S is moving with respect to the medium (air), & the receiver R is stationary, $\boxed{\text{OR}}$ (b) the source S is stationary and R is moving away. You find:

NB: problem for HW1

(a) $\nu' = \dfrac{\nu_0}{1 + \beta}$ $\qquad$ ν' = observed frequency $\quad \beta \equiv \dfrac{V}{c_s} \leftarrow$ sound speed

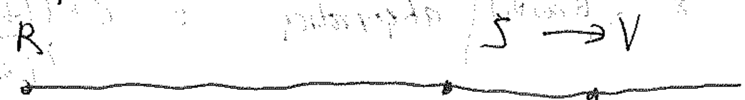
ν_0 = rest frequency

"redshift" S and R are receding

(b) $\nu' = \nu_0 (1 - \beta)$

Doppler Shift: Light

This is much simpler, since there is no medium: there is only one formula, depending on the relative velocity of S and R.



$t_1 \qquad t_2 \qquad t_2 - t_1 = \gamma \tau_0$

$x_1 \qquad x_2 \qquad x_2 - x_1 = V(t_2 - t_1) = V\gamma\tau_0$

Suppose the source S emits two brief pulses. In its rest frame, they are separated in time by τ_0 (proper time). So the observed time separation, to R, is $\boxed{\gamma \tau_0}$

The first pulse is received at R at time: $t_1 + x_1/c$

The 2nd pulse " " " " $t_2 + x_2/c$

So the observed time separation at R is $t_2 - t_1 + \frac{1}{c}(x_2 - x_1) \equiv \tau'$

Thus, $$\tau' = \gamma\tau_0 + \frac{V\gamma\tau_0}{c} = \gamma\tau_0\left(1 + \frac{V}{c}\right) = \frac{\tau_0 (1+\beta)}{\sqrt{1-\beta^2}} = \frac{\tau_0(1+\beta)}{\sqrt{1-\beta}\sqrt{1+\beta}} = \tau_0 \sqrt{\frac{1+\beta}{1-\beta}}$$

To find the observed frequency, use $\nu = 1/\tau$ →

$$\boxed{\nu' = \nu_0 \sqrt{\frac{1-\beta}{1+\beta}}} \; *$$

Finally the observed wavelength $(\lambda = c/\nu)$ is

$$\boxed{\lambda' = \lambda_0 \sqrt{\frac{1+\beta}{1-\beta}}}$$

Definition of "redshift"

$$z \equiv \frac{\lambda' - \lambda_0}{\lambda_0} = \frac{\Delta\lambda_0}{\lambda_0}$$

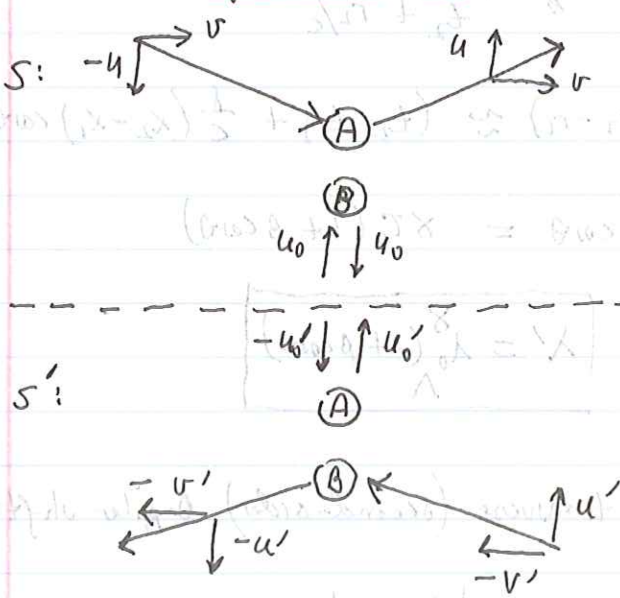
$$= \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

in this case*

Momentum

This will not be $\vec{p} = m\vec{v}$, but a generalization: $\vec{p} = \overbrace{m(v)}^{\text{"relativistic mass"}}\,\vec{v}$.

Consider a glancing collision of particles A & B with identical mass:



In S, B goes up & down w/ slow speed, $u_0 \lll c$, but v is big.

Transform to S′, where A goes up & down. This frame must move at speed V with respect to S.

Consider x-velocity of B before the collision. Applying the velocity transformation, and noting that the x-velocity is 0 in S,

$$-v' = \frac{0 - V}{1 - \frac{V}{c^2}\cdot 0} \rightarrow \boxed{V' = V}$$

Clearly, the picture in S′ is just a 180° flip of that in S. Erase all primes.

Now consider the y-velocity of B before the collision.

$$u = \frac{u_0}{\gamma\left(1 - \frac{V}{c^2}\cdot 0\right)} \rightarrow \boxed{u = u_0/\gamma}$$

Since $u_0 \lll c$, it is also true that $u \ll c$. So the speed of A in S is very close to v.

Now invoke conservation of y-momentum in frame S:

$$\underbrace{-m(V)u}_{A} + \underbrace{\overset{\text{rest mass}}{m_0} u_0}_{B} = \underbrace{+m(V)u}_{A} \underbrace{- m_0 u_0}_{B}$$

before collision   after collision

$$2m(V)u = 2m_0u_0 \rightarrow \frac{m(V)}{m_0} = \frac{u_0}{u} = \gamma \rightarrow \boxed{m(V) = \gamma m_0}$$

$$\boxed{\vec{p} = m\gamma\vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}} \quad \text{relativistic momentum}$$

Note that the "relativistic mass" increases with speed, explaining why objects cannot be accelerated indefinitely.

Energy

We have: $p = \gamma m v = m\gamma\beta c$

Also: $\gamma^2 = \frac{1}{1-\beta^2} \rightarrow \gamma^2 - \beta^2\gamma^2 = 1$ Now multiply through by m^2c^4

$$\gamma^2 m^2 c^4 - \underbrace{m^2\gamma^2\beta^2c^4}_{p^2c^2} = m^2c^4$$

Now pc has the units of energy. We define the total relativistic energy as

$$\boxed{E = \gamma mc^2}$$

Then, $\boxed{E^2 = p^2c^2 + m^2c^4}$

The last term on the RHS is defined as the square of the "rest energy" E_0

$$\boxed{E_0 = mc^2}$$

The relativistic kinetic energy is then $k \equiv E - mc^2$:

$$\boxed{k = mc^2(\gamma - 1) = mc^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)}$$

Thus, $v^2 = \dfrac{\dfrac{2k}{m} + \dfrac{k^2}{m^2c^2}}{\left(1 + \dfrac{k}{mc^2}\right)^2}$ (giving the $v^2(k)$ curve in L1)

In the non-relativistic ($\beta \ll 1$) limit, $\dfrac{1}{\sqrt{1-\beta^2}} \approx \dfrac{1}{1-\beta^2/2} \approx 1 + \beta^2/2$

so, $k \to mc^2\left(\dfrac{\beta^2}{2}\right) = \frac{1}{2}mv^2$ ✓

Finally, we can recover v from E and p: $\boxed{v} = \dfrac{p}{m\gamma} = \boxed{\dfrac{pc^2}{E}}$

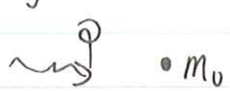
Application: Photon

A photon has zero rest mass, so $E = pc$ as before

It follows that $v = \dfrac{pc^2}{E} = c \rightarrow$ Any particle with $m = 0$ must move with $v = c$ in any reference frame.

Application: Absorption of Photon by Atom

A stationary atom of (rest) mass m_0 is struck by a photon of energy φ which is absorbed. What is new mass m and velocity v of the atom?



Energy conservation

$$m_0c^2 + \varphi = m\gamma c^2 \rightarrow m\gamma = m_0 + \frac{\varphi}{c^2}$$

NOT m_0!

Momentum conservation

$$\frac{\varphi}{c} = m\gamma v \rightarrow v = \frac{\varphi}{c(m\gamma)} = \frac{\varphi/c}{m_0 + \varphi/c^2}$$

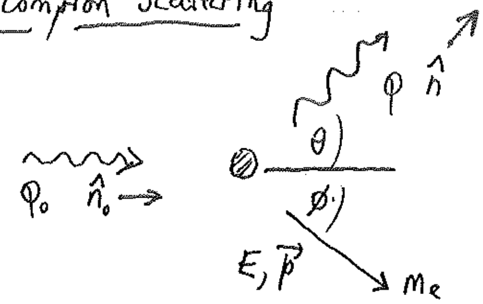
or $\boxed{\beta \equiv \dfrac{v}{c} = \dfrac{\varphi}{m_0c^2 + \varphi}}$

* Can show that the mass is now increased to

$$\boxed{\frac{m}{m_0} = \sqrt{1 + \frac{2\varphi}{m_0c^2}}}$$

Addendum: Compton Scattering

A photon strikes a stationary electron. The electron recoils at an angle ϕ. The photon recoils at an angle θ. Because the photon gives up some energy to the electron, its final energy is **less**. But photon's have $E = h\nu$. So the photon λ *increases* after scattering. We will find $\Delta\lambda$.



NB: $\hat{n}\cdot\hat{n}_0 = \cos\theta$

This process and its inverse (moving electron *adding energy* to a photon) are important in astrophysics

Energy conservation: $\quad \varphi_0 + m_e c^2 = \varphi + E$

Mom. conservation: $\quad \dfrac{\varphi_0}{c}\hat{n}_0 = \dfrac{\varphi}{c}\hat{n} + \vec{p}$

Thus
$$(\varphi_0 - \varphi) + m_e c^2 = E$$
$$(\varphi_0 \hat{n}_0 - \varphi\hat{n}) = \vec{p}c$$

Now square both equations.

$$\varphi_0^2 - 2\varphi_0\varphi + \varphi^2 + 2(\varphi_0 - \varphi)m_e c^2 + \cancel{m_e^2 c^4} = E^2 = p^2c^2 + \cancel{m_e^2 c^4}$$
$$\varphi_0^2 - 2\varphi_0\varphi\cos\theta + \varphi^2 = p^2c^2$$

where I have used $\hat{n}_0\cdot\hat{n} = \cos\theta$

Subtract:
$$-2\varphi_0\varphi(1-\cos\theta) + 2(\varphi_0 - \varphi)m_e c^2 = 0 \qquad \text{Multiply by } \frac{1}{2\varphi_0\varphi\, m_e c^2}$$

$$-\frac{(1-\cos\theta)}{m_e c^2} + \frac{(\varphi_0 - \varphi)}{\varphi_0\varphi} = 0$$

$$\frac{1}{\varphi} - \frac{1}{\varphi_0} = \frac{(1-\cos\theta)}{m_e c^2} \qquad \text{But } \varphi = h\nu = \frac{hc}{\lambda} \rightarrow \frac{1}{\varphi} = \frac{\lambda}{hc}$$

$$\frac{\lambda}{hc} - \frac{\lambda_0}{hc} = \frac{(1-\cos\theta)}{m_e c^2} \longrightarrow \boxed{\Delta\lambda = \lambda - \lambda_0 = \frac{h}{m_e c}(1-\cos\theta)}$$

$\dfrac{h}{m_e c}$ ($= 0.024$ Å) is the electron's "Compton wavelength"