L22 - Newtonian Cosmology

Evolution of $R(t)$

Our first goal is to ascertain the evolution of the scale factor $R(t)$. [Recall: $R(t_0) = 1$]

So far, we have $\boxed{v^2 - \frac{8\pi}{3} G\rho r^2 = -kc^2\varpi^2}$ energy equation

Now $r(t) = R(t)\varpi$, so $r^2 = R^2\varpi^2$

$v = \frac{dr}{dt} = \frac{dR}{dt}\varpi$, so $v^2 = \left(\frac{dR}{dt}\right)^2\varpi^2$

Dividing through by ϖ^2, we have an equation for R, not r:

$$\boxed{\left[\frac{1}{R}\left(\frac{dR}{dt}\right)^2 - \frac{8\pi}{3}G\rho\right]R^2 = -kc^2}$$

Friedman equation
$k =$ constant

This equation turns out to be generally true. For our dusty universe,

$\boxed{\rho = \rho_0/R^3}$ and the equation becomes

$$\boxed{\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho_0}{3R} = -kc^2}$$

Friedman eqtn for dust

Critical Density

Is Hubbles law obeyed in this model?
Yes!

$$r(t) = R(t)\varpi$$

$$v = \frac{dr}{dt} = \frac{dR}{dt}\varpi = \underbrace{\left(\frac{1}{R}\frac{dR}{dt}\right)}_{\text{indpt of } \varpi} R\varpi = \left(\frac{1}{R}\frac{dR}{dt}\right) r$$

Thus, $\boxed{H = \frac{1}{R}\frac{dR}{dt}}$

Suppose that $k=0$ (flat Universe). Then $\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \frac{8\pi G\rho}{3} = H^2$

In this, ρ equals the critical density $\boxed{\rho_c \equiv \frac{3H^2}{8\pi G}}$

dimensionally ok, since $\frac{1}{H^2} \sim t^2 \sim \frac{1}{\rho G}$ ✓

To evaluate ρ_c today, we use $H_0 = 71$ km s^{-1} Mpc^{-1} to find

$$\boxed{\rho_c = 9.5 \times 10^{-27} \text{ kg m}^{-3}}$$ 6 H atoms per m^3

The best estimate of the actual density of [baryonic] matter is 4% of this! Here we are counting up galaxies + applying $\langle L/M \rangle$.

The density parameter for baryons is defined as

$$\boxed{\Omega_b \equiv \frac{\rho_b}{\rho_c} = \frac{8\pi G \rho_b}{3H^2}}$$

The best current estimate for the total matter density parameter, including DM, is

$$\boxed{\Omega_m = \frac{\rho_m}{\rho_c} = 0.27}$$

Relation between Ω_0 + k

From $\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}\rho\right] R^2 = -kc^2$, we have

$$\left(H^2 - \frac{8\pi G \rho}{3}\right) R^2 = -kc^2$$

Write $\rho = \Omega \rho_c$

so $\frac{8\pi G \rho}{3} = \Omega\left(\frac{8\pi G \rho_c}{3}\right) = \Omega H^2$

$$(1) \quad \boxed{H^2(1-\Omega) R^2 = -kc^2}$$

Today we have

$$\boxed{H_0^2(1-\Omega_0) = -kc^2}$$ since $R(t_0) = 1$

So k is given by the magnitude of today's H and Ω. We see again that

$\Omega_0 = 1 \longrightarrow k = 0$

—and—

$\Omega_0 < 1 \longrightarrow k < 0$ open universe

$\Omega_0 > 1 \longrightarrow k > 0$ closed universe

Evolution of Ω: Flatness Problem

Eqtn (1) is one relation between Ω and H, together with R. Let's replace R by z, using $R = (1+z)^{-1}$. Then (1) becomes

$$H^2(1-\Omega)R^2 = -kc^2 = H_0^2(1-\Omega_0)$$

$$(2) \quad \boxed{H^2(1-\Omega) = H_0^2(1-\Omega_0)(1+z)^2}$$

To get another relation, we use $\rho = \frac{\rho_0}{R^3} = \rho_0(1+z)^3$. Then:

$$\frac{\Omega}{\Omega_0} = \frac{\rho}{H^2}\frac{H_0^2}{\rho_0} = \frac{\rho}{\rho_0}\frac{H_0^2}{H^2} = \frac{H_0^2}{H^2}(1+z)^3 \quad \text{so that}$$

$$(3) \quad \boxed{\Omega H^2 = \Omega_0 H_0^2 (1+z)^3}$$

$$(2)/(3) \rightarrow \boxed{\left(\frac{1}{\Omega}-1\right) = \left(\frac{1}{\Omega_0}-1\right)\frac{1}{1+z}} \longrightarrow \boxed{\left(\frac{1}{\Omega_0}-1\right) = \left(\frac{1}{\Omega}-1\right)(1+z)}$$

- If $\Omega < 1$ before, $\Omega_0 < 1$ now. Open evolves to open, closed to closed, etc
- Whatever Ω is now, it was very close to 1 when $z \gg 1$. So the very early Universe was essentially flat.
- Conversely, if Ω differed even slightly from 1 in the past, it would differ hugely from 1 now

Why should Ω_0 be even roughly 1 now? This is the flatness problem.

Evolution of H

Let's solve the penultimated boxed eqtn explicitly for Ω:

$$\frac{1}{\Omega} = 1 + \frac{\frac{1}{\Omega_0}-1}{1+z} = \frac{1+z+\frac{1}{\Omega_0}-1}{1+z} = \frac{z+\frac{1}{\Omega_0}}{1+z} = \frac{(1+\Omega_0 z)}{\Omega_0(1+z)}$$

so that $\Omega = \frac{\Omega_0(1+z)}{1+\Omega_0 z}$ Now plug this into (2) to find $H(t)$.

$$1-\Omega = 1 - \frac{\Omega_0(1+z)}{1+\Omega_0 z} = \frac{1+\Omega_0 z - \Omega_0 - \Omega_0 z}{1+\Omega_0 z} = \frac{1-\Omega_0}{1+\Omega_0 z}$$

Using eqtn (2), $\frac{H^2(1-\Omega_0)}{1+\Omega_0 z} = H_0^2(1-\Omega_0)(1+z)^2$

$$H^2 = H_0^2(1+z)^2(1+\Omega_0 z)$$

$$\boxed{H = H_0(1+z)(1+\Omega_0 z)^{1/2}}$$

In the past, H was larger than today. Expansion was faster.

Evolution of R(t)

We have: $\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho_0}{3R} = -kc^2 = \left(\frac{dR}{dt}\right)_0^2 - \underbrace{\left(\frac{8\pi G\rho_0}{3}\right)}_{\frac{\Omega_0 H_0^2}{R}} = H_0^2(1-\Omega_0)$

Define $\tau \equiv H_0 t$ and divide through by H_0^2

$$\boxed{\left(\frac{dR}{d\tau}\right)^2 - \frac{\Omega_0}{R} = 1-\Omega_0}$$

Can be solved for any Ω_0.

We will only solve this equation for $\Omega_0 = 1$.

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{1}{R} \longrightarrow R^{1/2}\frac{dR}{d\tau} = 1$$

$$\frac{2}{3}R^{3/2} = \tau \longrightarrow R = \left(\frac{3}{2}\right)^{2/3}\tau^{2/3}$$

$$\boxed{R(t) = \left(\frac{3}{2}\right)^{2/3}(H_0 t)^{2/3}}$$

$\Omega_m = 1$ universe

In this model, for any z, the corresponding t is

$$t = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}}$$

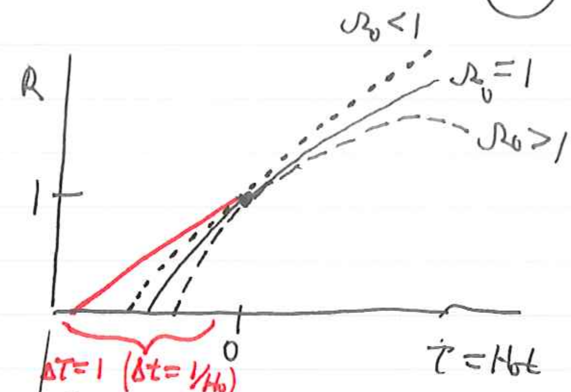
eg when $z=2$, $t = \frac{t_0}{2^{3/2}} = \frac{1}{4}t_0$!

To find the age of the Universe, set $R=1$:

$$t_0 = \frac{2}{3H_0} = 9.2 \text{ Gyr} \quad \text{for } h = 0.71$$

> This cannot be right, since there are older globular clusters!

When displaying solutions, it is convenient to reset time so that $\tau = 0$ now. Then we demand $R = 1$ and $dR/d\tau = 1$ at $\tau = 0$.



Example – When $\Omega_0 = 0.5$, age of the Universe is 10 Gyr, etc

NB: t_0 is always $< H_0^{-1}$

Including Pressure

If the matter has pressure, it is not true that $\rho \propto 1/R^3$.

another heuristic derivation

The density ρ is associated w/ an energy density ρc^2 [sneaking in relativity!]
The total energy in a (filled) sphere is

$$U = \frac{4\pi}{3} \rho c^2 R^3$$

First law of thermodynamics $\Delta U = -P\Delta V = -P \frac{4\pi}{3} \Delta R^3$ (adiabatic)

$$\boxed{\frac{d}{dt}(\rho R^3) = -\frac{P}{c}\frac{dR^3}{dt}}$$

$\rho \propto R^{-3}$ only if $P = 0$

We now derive an eqn for the acceleration d^2R/dt^2. Start with

$$R\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho R^3}{3} = -kc^2 R \qquad \text{Take } \frac{d}{dt}:$$

$$\cancel{\left(\frac{dR}{dt}\right)^3} + 2R\frac{d^2R}{dt^2}\frac{dR}{dt} + \frac{8\pi G}{3}\frac{P}{c^2}\frac{dR^3}{dt} = -kc^2\frac{dR}{dt}$$

$$= \cancel{\left(\frac{dR}{dt}\right)^3} - \frac{8\pi G\rho}{3}R^2\frac{dR}{dt}$$

Dividing by $2R\, dR/dt$,

$$\frac{d^2R}{dt^2} + \frac{8\pi G}{3}\frac{P}{c^2}\frac{3R^2\, dR/dt}{2R\, dR/dt} = -\frac{4\pi G\rho}{3}R$$

Two basic equations of cosmology

$$\boxed{\frac{d^2R}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)R}$$ acceleration equation

$$\boxed{\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho}{3}R^2 = -kc^2}$$ Friedman (energy) equation

NB Higher (positive) P goes R(t) to slow down!