L23 — Relativistic Cosmology: I

Robertson-Walker Metric

Recall that, in SR, the spacetime interval is

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

so that
$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

We could also write the interval in spherical coordinates (r, θ, φ):

$$\Delta s^2 = (c\Delta t)^2 - (\Delta r)^2 - (r\Delta\theta)^2 - (r\sin\theta\,\Delta\varphi)^2$$

Here
$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -r^2 & \\ 0 & & & -r^2\sin^2\theta \end{pmatrix}$$

Both metrics describe flat spacetime (Minkowski), i.e., the curvature (obtained using the metric) is zero.

Recall that the presence of matter curves spacetime, producing metrics that are not flat. We studied only one, the Schwarzschild metric describing spacetime surrounding a static, uncharged point mass:

$$(\Delta s)^2 = \left(c\sqrt{1 - 2GM/rc^2}\,\Delta t\right)^2 - \left(\frac{\Delta r}{\sqrt{1 - \frac{2GM}{rc^2}}}\right)^2 - (r\Delta\theta)^2 - (r\sin\theta\,\Delta\varphi)^2$$

$$g_{\mu\nu} = \begin{pmatrix} (\sqrt{\ })^2 & & & 0 \\ & -\left(\frac{1}{\sqrt{\ }}\right)^2 & & \\ & & -r^2 & \\ 0 & & & -r^2\sin^2\theta \end{pmatrix}$$

This metric is a solution to the Einstein field equations

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \qquad \text{with } T_{\mu\nu} = 0$$

Robertson + Walker, in the 1930s, first found the general $g_{\mu\nu}$ in spherical coordinates for a spacetime that is homogeneous and isotropic:

$$\boxed{ds^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1 - kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\varphi)^2}$$

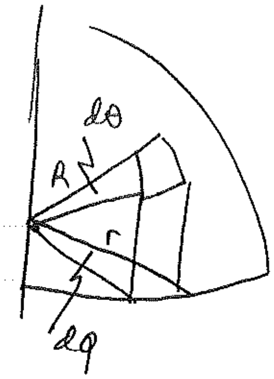
Here, k is the curvature. This is the metric used to describe the Universe in GR.

Aside: Geometric Interpretation

The spatial part of $g_{\mu\nu}$ was inspired by non-Euclidean geometry, developed in the 19th century

The distance between 2 points on a sphere of radius R is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2$$



We can write this in terms of coordinates (r, φ) in the equatorial plane.

Let $r = R\sin\theta$

$$\rightarrow dr = R\cos\theta \, d\theta \quad \longrightarrow \quad \frac{dr}{\cos\theta} = R\, d\theta = \frac{dr}{\sqrt{1-\sin^2\theta}} = \frac{dr}{\sqrt{1 - r^2/R^2}}$$

Thus $ds^2 = \left(\dfrac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + r^2 d\varphi^2$ in plane, polar coordinates

We want to generalize the above to represent a distance in a 3D curved space, which is hard to visualize. Every 2D slice of the 3D space should be a curved, 2D space. The answer, worked out by the mathematicians, is to replace $d\varphi^2$ above by the [square of the] small angular displacement in spherical coordinates that is $\perp$ to the r-displacement:

$$d\varphi^2 \longrightarrow d\theta^2 + \sin^2\theta \, d\varphi^2$$

So the metric for a 3D space w/ radius of curvature R is

$$\boxed{ds^2 = \left(\frac{dr}{\sqrt{1 - \frac{r^2}{R^2}}}\right)^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

Let $K \equiv \frac{1}{R^2}$, the Gaussian curvature. This measures how much a vector changes if transported around a closed loop. [Rather – if you transport the vector along geodesics, by how much does the total angle turned in a closed loop differ from 360°?]

If $K = 0$, we get a flat, 3D geometry (Euclidean)

$K > 0$ closed ~~space~~ geometry ("elliptic") ∃ no parallel lines

$K < 0$ open geometry ("hyperbolic") ∃ ∞ no. of parallel lines

Back to Robertson-Walker

For 2 events separated by a spacelike interval, it may be that $dt = 0$. Then the "proper distance" between the events,

$$\Delta \ell = \sqrt{\left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2}$$

may change with time, if $K = K(t)$. In fact, the proper distance between galaxies does change w/ time. Usually, $\Delta\ell$ means the proper distance at the time the galaxy emitted the light we see today.

We define a comoving radial coordinate ϖ through $r = R\varpi$ where $R(t)$ is a dimensionless scale factor that is unity now. The R-W metric becomes

$$\boxed{ds^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]}$$

Here, $k \equiv R^2 K$ is the curvature constant.

> The comoving distance to a galaxy is the proper distance at the present time, when $R = 1$, assuming the galaxy followed the Hubble flow (no peculiar velocity).

Evolutionary Equations

Einstein modeled the matter in the Universe as a perfect fluid. That is,

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & -p & & \\ & & -p & \\ 0 & & & -p \end{pmatrix}$$

for a fluid of mass density ρ and pressure P.

Using this $T_{\mu\nu}$, Friedmann found an evolutionary eqn for $R(t)$

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho}{3}\right]R^2 = -k^2c^2 \qquad \text{Friedmann equation}$$

together with

$$\frac{d}{dt}\left(\rho R^3\right) = -\frac{P}{c^2}\frac{dR^3}{dt} \qquad \text{fluid equation}$$

These are the same as in Newtonian cosmology. Thus —

Combining the Friedmann and acceleration equation yields, as before,

$$\boxed{\frac{d^2R}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)R}$$

acceleration equation

Equation of State

To find $R(t)$, we need the relation between P and ρ, and the functional dependence of ρ on R.

For dust, $P = 0$ and $\rho = \rho_0 R^{-3}$.

In general write

$$P = \omega \rho c^2$$

[dust $\rightarrow \omega = 0$] [radiation $\rightarrow \omega = \frac{1}{3}$]

The fluid eqtn tells us $\dfrac{d(\rho R^3)}{dt} = -\dfrac{P}{c^2}\dfrac{dR^3}{dt}$

$$d(\rho R^3) = d\rho\, R^3 + \rho\, dR^3 = -\omega \rho\, dR^3$$

or

$$R^3 d\rho + (1+\omega)\rho\, dR^3 = 0 \rightarrow \frac{d\rho}{\rho} + (1+\omega)\frac{dR^3}{R^3} = 0$$

Integrating,

$$\ln\rho + (1+\omega)\ln R^3 = \ln\rho + \ln R^{3(1+\omega)} = \text{const}, \text{ so that}$$

$$\boxed{\rho R^{3(1+\omega)} = \text{const}}$$

eg when radiation dominated, $\rho R^4 = $ constant

Cosmological Constant

Einstein (1917) modified his field equation to

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu}}$$

$\Lambda =$ cosmological constant

units: R^{-2}

He introduced Λ since, without it, $R(t)$ could not be constant. After Hubble's 1929 discovery, he called Λ a huge mistake. It is now back in favor.

The Friedmann equation becomes

$$\boxed{\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho}{3} - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2}$$

Friedmann

while the fluid equation is still

$$\boxed{\frac{d(\rho R^3)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}}$$

fluid

Combining Friedmann and fluid yields the acceleration equation*

$$\boxed{\frac{d^2R}{dt^2} = \left[-\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}\right]R}$$ acceleration eq'n

Note that the effect of Λ is to speed up the acceleration of R. As we will see, this seems to be called for observationally.* $\boxed{\text{This is "}\Lambda\text{CDM" cosmology.}}$

Pressure of Dark Energy

We may associate an equivalent density ρ_Λ as follows: (motivated by Friedmann equation)

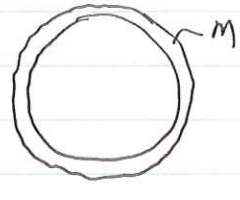
$$\frac{8\pi G}{3}\rho_\Lambda = \frac{1}{3}\Lambda c^2 \longrightarrow \boxed{\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant}}$$

Find the equivalent pressure from the fluid equation:

$$\rho_\Lambda \frac{dR^3}{dt} = -\frac{P_\Lambda}{c^2}\frac{dR^3}{dt} \longrightarrow \boxed{P_\Lambda = -\rho_\Lambda c^2} \quad (w = -1)$$

Dark energy has negative pressure!

Equivalent Newtonian Potential



Starting with Friedmann in the form

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}\rho R^2 - \frac{1}{3}\Lambda c^2 R^2 = -kc^2$$

multiply by $\frac{1}{2}m\varpi^2$, where m is the mass of the Newtonian shell

$$\frac{1}{2}m\varpi^2\left(\frac{dR}{dt}\right)^2 - \frac{4\pi G}{3}\rho m \varpi^2 R^2 - \frac{1}{6}\Lambda c^2 m\varpi^2 R^2 = \underbrace{-\frac{k}{2}c^2 m\varpi^2}_{E\ =\ \text{constant}}$$

Recognizing $\varpi R = r$, and using $v = dr/dt$,

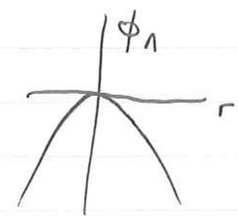
$$\frac{1}{2}mv^2 - \frac{4\pi}{3}\frac{G\rho r^3}{r}m - \frac{1}{6}\Lambda c^2 m r^2 = E$$ But $\frac{4\pi}{3}\rho r^3 = M_r$

$$\frac{1}{2}mv^2 - \frac{GM_r}{r}m - \frac{\Lambda c^2 r^2}{6}m = E$$

Newtonian potential — dark energy potential

(ϕ_Λ)

$\boxed{\text{The potential is } -\frac{\Lambda c^2 r^2}{6} \text{ — repulsive}}$



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* In "de Sitter model," $P = \rho = 0$, and $\Lambda \neq 0$. Universe accelerates in an expanding fashion. $\Lambda < 0$ is known as "anti-de Sitter space."