L24 - Relativistic Cosmology: II

Density Parameters and History of H

Recall that $H = \frac{1}{R}\, dR/dt$ at any time, and that the critical density at that time is $\rho_c = 3H^2/8\pi G$. Rewrite Friedmann equation as

$$\left[H^2 - \frac{8\pi G\rho}{3} - \frac{1}{3}\Lambda c^2\right]R^2 = H^2\left[1 - \frac{8\pi G\rho}{3H^2} - \frac{\Lambda c^2}{3H^2}\right]R^2 = -kc^2$$

or

$$\boxed{H^2\left(1 - \Omega_m - \Omega_\Lambda\right)R^2 = -kc^2}$$

relation of H & Ω's to k (const) at any epoch

where

$\Omega_m \equiv \rho/\rho_c$, the density parameter for matter (baryons plus dark matter)

and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$, " " " " dark energy. We may also add Ω_{rel}, the parameter for relativistic particles (with $\omega = 1/3$).

We then have

$$H^2(1 - \Omega_m - \Omega_{rel} - \Omega_\Lambda)R^2 = \underbrace{H_0^2(1 - \Omega_{m,0} - \Omega_{rel,0} - \Omega_{\Lambda,0})}_{\text{Values today}}$$

$$= H_0^2(1 - \Omega_0) = -kc^2$$

To find the history of H, ie, $H(z)$, we need to know $\Omega_m(z)$, etc.

We have

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m}{H^2} \cdot \frac{H_0^2}{\rho_{m,0}} = \frac{\rho_m}{\rho_{m,0}}\frac{H_0^2}{H^2} = \frac{H_0^2}{H^2}(1+z)^3$$

since $\rho \propto R^{-3}$ for $\omega = 0$

$$\frac{\Omega_{rad}}{\Omega_{rad,0}} = \frac{\rho_{rel}}{\rho_{rel,0}}\frac{H_0^2}{H^2} = \frac{H_0^2}{H^2}(1+z)^4$$

since $\rho \propto R^{-4}$ for $\omega = 1/3$

$$\frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{\rho_\Lambda}{\rho_{\Lambda,0}}\frac{H_0^2}{H^2} = \frac{H_0^2}{H^2}$$

since $\rho_\Lambda =$ constant

Using these ratios, plus $R = (1+z)^{-1}$, algebraic manipulation* yields

$$\boxed{H = H_0(1+z)\left[\Omega_{m,0}(1+z) + \Omega_{rel,0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0\right]^{1/2}}$$

Compare this to Newtonian result -

$$H = H_0(1+z)\left[1 + \Omega_0 z\right]^{1/2}$$

which we recover if $\Omega_{m,0} = \Omega_0$ and $\Omega_{rel,0} = \Omega_{\Lambda,0} = 0$

We will use $H(z)$ to find the important distance-magnitude relation.
Currently accepted Ω-values are

$$\Omega_{m,0} = 0.27\,, \quad \Omega_{rel,0} = 8\times10^{-5}\,, \quad \Omega_{\Lambda,0} = 0.73$$

so $\Omega_0 = 1$ $(k=0)$.

The Future of the Universe

The full acceleration equation is now

$$\frac{d^2R}{dt^2} = \left[-\frac{4\pi G}{3}\left(\rho_m + \rho_{rel} + \frac{3P_{rad}}{c^2}\right) + \frac{\Lambda c^2}{3}\right]R$$

Both ρ_m and ρ_{rel} will decline in the future, but Λ will be constant.
We will then have

$$\frac{d^2R}{dt^2} \doteq \frac{\Lambda c^2}{3} R$$

dimensional check!
$\Lambda c^2 R \sim \frac{1}{R^2}\cdot R\cdot c^2 \sim R/t^2$ ✓

so that $R \propto \exp\sqrt{\frac{\Lambda c^2}{3}}\,t$

But

$$\frac{\Lambda c^2}{3} = \frac{\Lambda c^2}{8\pi G}\cdot\frac{8\pi G}{3} = \rho_\Lambda\cdot\frac{8\pi G}{3} = \rho_\Lambda H_0^2\cdot\underbrace{\frac{8\pi G}{3H_0^2}}_{\rho_c^{-1}}$$

$$= \frac{\rho_\Lambda}{\rho_c}H_0^2 = \Omega_\Lambda H_0^2$$

That is, $\boxed{R \propto \exp\left[\sqrt{\Omega_\Lambda}\,(H_0 t)\right]}$

The e-folding time is

$$\frac{1}{\sqrt{\Omega_\Lambda}}H_0^{-1} = 1.17\,H_0^{-1}$$

(using $\Omega_\Lambda = 0.73$)

The Λ Era

We switched over to being Λ-dominated when $\boxed{\rho_m = \rho_\Lambda}$, or when

$$\underbrace{\rho_m R^3}_{\rho_{m,0}} = \rho_\Lambda R_{m\Lambda}^3 \longrightarrow \frac{\rho_{m,0}}{\rho_{c,0}} = \frac{\rho_\Lambda}{\rho_{c,0}}R_{m\Lambda}^3$$

$R_{m\Lambda}$ = scale factor at switchover point

$$\Omega_{m,0} = \Omega_{\Lambda,0}\,R_{m\Lambda}^3$$

$$R_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = 0.72$$

The Universe was about ½ its present age.
Prior to that time, $R(t)$ was decelerating.

Full Evolution of $R(t)$

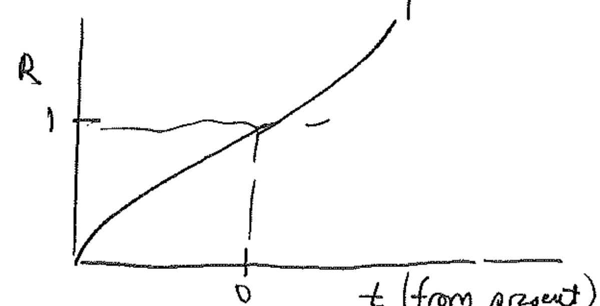
We could integrate the acceleration equation, but it is easier to set $k=0$ in the Friedmann equation:

$$\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)$$
$$\doteq \frac{8\pi G}{3}\left(\rho_{m,0} R^{-3} + \rho_{rel,0} R^{-4} + \rho_{\Lambda,0}\right)$$

so that $$\frac{R\,dR}{dt} = \sqrt{\frac{8\pi G}{3}}\left(\rho_{m,0} R + \rho_{rel,0} + \rho_{\Lambda,0} R^4\right)^{1/2}$$

$$\boxed{t = \int dt = \sqrt{\frac{3}{8\pi G}}\int_0^R \frac{R\,dR}{\sqrt{\rho_{m,0} R + \rho_{rel,0} + \rho_{\Lambda,0} R^4}}}$$

Plug in values of $\rho_{m,0}$; $\rho_{rel,0}$, and ρ_Λ and integrate numerically



Putting the upper limit $R=1$, we find $\boxed{t_0 = 13.7\text{ Gyr}}$

Coincidentally, this is close to H_0^{-1}!

Origin of Cosmic Redshift

For a photon, $ds=0$, so the R-W metric gives

$$\frac{c\,dt}{R(t)} = \pm\frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

for a photon moving radially ($d\theta = d\phi = 0$)

A galaxy with $\varpi = \varpi_{em}$ emits the 1st crest of a lightwave at $t = t_{em}$. This crest is received by us ($\varpi = 0$) at $t = t_{rec}$. Choosing the $\ominus$ sign above,

$$c\int_{t_{em}}^{t_{rec}} \frac{dt}{R(t)} = -\int_{\varpi_{em}}^{0}\frac{d\varpi}{\sqrt{1-k\varpi^2}} = +\int_0^{\varpi_{em}}\frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

The second crest is emitted at $t = t_{em} + \delta t_{em}$, + received at $t = t_{rec} + \delta t_{rec}$

The ϖ-coordinates are unchanged — ϖ_{em} and 0.

$$\int_{t_{em}+\delta t_{em}}^{t_{rec}+\delta t_{rec}} \frac{dt}{R(t)} = \cancel{\int_{t_{em}}^{t_{rec}} \frac{dt}{R(t)}} = \int_{t_{em}+\delta t_{em}}^{t_{em}} + \cancel{\int_{t_{em}}^{t_{rec}}} + \int_{t_{rec}}^{t_{rec}+\delta t_{rec}}$$

During both Δt_{em} + Δt_{rec}, $R(t)$ does not change appreciably. We thus have

$$\boxed{\frac{\Delta t_{em}}{R(t_{em})} = \frac{\Delta t_{rec}}{R(rec)}}$$

True for any temporal process ("cosmological time dilation")

For the light wave, use $R(rec)=1$, $\Delta t_{em} = \frac{\lambda_{em}}{c}$, $\Delta t_{rec} = \frac{\lambda_{rec}}{c}$, so

$$\boxed{\lambda_{rec} = \frac{\lambda_{em}}{R(t_{em})} = \lambda_{em}(1 + z_{em})}$$

Horizon Distance

As time goes on, photons arrive from ever more distant objects. The farthest observable point is located at $d_H(t_0)$, the horizon distance. Points separated by more than $d_H(t_0)$ have never been in causal contact.

Formally, $d_H(t_0)$ is the proper distance today to a galaxy whose emitted photon is just reaching us now.

$$d\ell = \underset{1}{R(t_{rec})} \frac{d\varpi}{\sqrt{1-k\varpi^2}} \longrightarrow d_H(t_0) = \int_0^{\varpi_{em}} \frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

What is ϖ_{em}? Let the photon be emitted at $t=0$ (Big Bang). Then, since $ds=0$,

$$\int_0^{\varpi_{em}} \frac{d\varpi}{\sqrt{1-k\varpi^2}} = \int_0^{t_0} \frac{c\,dt'}{R(t')}$$

Thus $\boxed{d_H(t_0) = \int_0^{t_0} \frac{c\,dt'}{R(t')}}$

Example - In an $\Omega_0 = 1$ Universe with $\Lambda = 0$,

$$R(t) = \left(\tfrac{3}{2}\right)^{2/3} (H_0 t)^{2/3}$$

So $R^{3/2} = \frac{3}{2} H_0 t \longrightarrow \frac{3}{2} R^{1/2} dR = \frac{3H_0}{2} dt \longrightarrow dt = R^{1/2}\left(\frac{1}{H_0}\right)$

$$d_H(t_0) = \frac{c}{H_0}\int_0^1 R^{-1/2} dR = \frac{2c}{H_0} = \frac{2\times 3\times 10^5 \text{ km/s}}{71 \text{ km s}^{-1} \text{ Mpc}^{-1}} = 8\times 10^3 \text{ Mpc}$$

Doing this more carefully, we find $\boxed{d_H(t_0) = 1.5\times 10^4 \text{ Mpc}}$

Redshift-Magnitude Relation

We may test cosmology models by observing the relationship between the apparent magnitude & redshift for "standard candles."

Naively, we expect $m - M = 5\log\left(\frac{d}{10\text{ pc}}\right)$ proper distance

To relate d to z, we (naively) use $V_r = H_0 d \rightarrow \frac{V_r}{c} = z = \frac{H_0}{c} d$
so $d = \frac{c}{H_0} z$. Using $H_0 = 100 \times h$,

$$m - M = 5\log\left(\frac{c}{100\ \mathrm{km\,s^{-1}\,Mpc^{-1}} \times 10\,\mathrm{pc}}\right) - 5\log h - 5\log z$$

naive ↗

In fact, this is only the first approximation. More carefully, the "d" in the distance modulus is the "luminosity distance" d_L: $\boxed{F = \frac{L}{4\pi d_L^2}}$ definition

But $\boxed{F = \frac{L}{4\pi \varpi^2 (1+z)^2}}$ One $(1+z)$ factor: photon redshift.
Second factor: decline in rate of photon reception (cosmological time dilation)

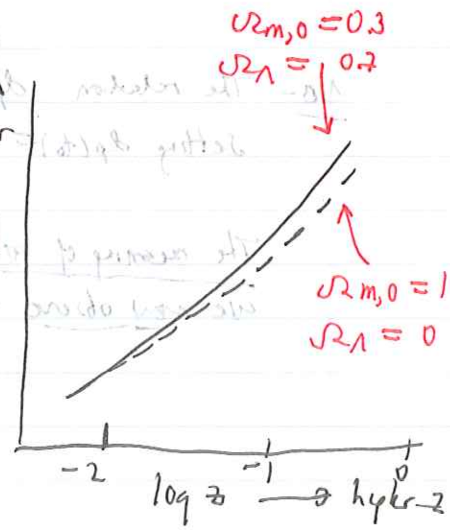
Thus $\boxed{d_L = \varpi(1+z)}$ → Need $\varpi(z)$

Finding $\varpi(z)$

Use two expressions for proper distance, and find* that $\varpi = c\int_0^z \frac{dz}{H(z)}$

As at the beginning of this lecture, we obtain $H(z)$ from knowledge of $\Omega_{m,0}$; $\Omega_{rel,0}$; and $\Omega_{\Lambda,0}$. Then use $\boxed{d_L = c(1+z)\int_0^z \frac{dz}{H(z)}}$

Result →

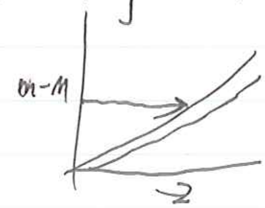


Type Ia Supernovae

In these supernovae, mass is transferred to a white dwarf. Once it exceeds the Chandrasekhar limit, the star collapses. Both empirically & theoretically, such supernovae appear to be a "standard candle".

Starting in ~2000, observers saw that the supernovae were too dim for their given z (obtained from spectral emission lines). The discrepancy, about 0.25 mag at $z \simeq 1$, seems to be fit by the $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ curve, but NOT by other cosmologies. The best evidence yet for accelerating $R(t)$

Qualitative account:



For a given apparent brightness (& thus d_L), the host galaxies move with too low a V_r. The expansion was too slow in the past, & has thus been accelerating