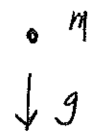
L3 - General Relativity I

GR is Einstein's theory of gravity, which superseded Newton's. Some essential features of GR are:

- Mass curves spacetime. Thus, GR is a <u>geometric</u> theory of gravity.
- Test particles move on the straightest possible trajectories within this curved spacetime. These trajectories appear to be orbits.
- Even light is thus bent in a gravitational field. — a new prediction.
- Light is also redshifted (λ increases) as it climbs out of a gravitational field.

Gravity as a Fictitious Force

Recall free-fall in Newtonian gravity. Galileo found that $a = g$ <u>regardless of the mass</u>. Newton's explanation was



$$F = mg = ma \longrightarrow a = g$$

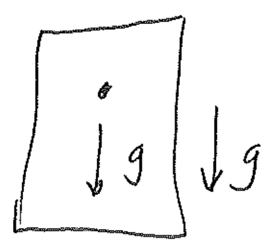
But actually, there are two different masses:

m_g - the gravitational mass that determines the force on the body

m_i - the inertial mass that gives the sluggishness with which the body responds to <u>any</u> force, gravity or not

So, to Newton, it was a coincidence that $m_g = m_i$

Einstein viewed the matter another way. If we are in a freely falling environment (an elevator), the object would appear to float. He called this <u>the principle of equivalence</u>:

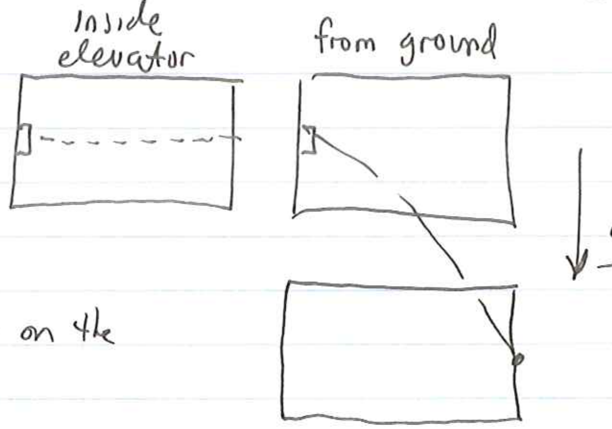


$\boxed{\text{In any inertial (freely falling) reference frame, gravity vanishes}}$

We recall that there are other forces that disappear when one is not in the right reference frame - centrifugal force, Coriolis force, etc. So one only detects gravity if one is in a non-inertial reference frame. In any inertial frame, the dynamics of <u>special relativity</u> apply.

Bending of Light

Consider shooting a flashlight beam within a falling elevator:



An observer inside the elevator sees no gravity. Thus, the beam travels in a straight line and hits the middle of the opposite wall.
The same must be true for an observer on the ground.
<u>Thus light bends gravity.</u>

<u>Curvature of Space</u>

Consider two closely spaced points in 3D space, separated by $\Delta x, \Delta y, \Delta z$
The distance between them is

"metric" $$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \equiv \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2$$

We can think of this as the dot product of $(\Delta x_1, \Delta x_2, \Delta x_3)$ with itself.
More formally

$$\Delta s^2 = \sum_{ij} \underbrace{h_{ij}}_{\text{"metric coefficients"}} \Delta x_i \Delta x_j$$

Here, $h_{ij} = \begin{matrix} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{matrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

Of course, we can only use another coordinate system. E.g. spherical:

$$\Delta s^2 = (\Delta r)^2 + (r\Delta\theta)^2 + (r\sin\theta\,\Delta\varphi)^2$$

The metric coefficients are now different, but Δs^2 is the same as long as the coordinates are related by the proper transformation. We are viewing the same line segment in two different coordinate systems. <u>The distance is invariant</u>

<u>Differential geometry</u>: We can construct a complicated function of the metric coefficients and their derivatives w/ respect to the coordinates called the "curvature tensor." In the above examples, all components of this tensor vanish, in any system. But if we draw a short arc on the surface of a sphere, the tensor would <u>not</u> vanish: space is curved. (eg the interior angles of a Δ do not sum to $180°$)

Curvature of Spacetime

Going back to special relativity, and an inertial frame, consider two closely spaced events. Construct the spacetime interval

$$\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$ flat-space metric

Which we can write as $\Delta s^2 = \eta_{ij} \Delta x^i \Delta x^j$ $\quad \Delta x^0 = c\Delta t$, $\Delta x^1 = \Delta x$ etc.

The metric coefficient are: $\eta_{ij} = \begin{cases} 1 & \text{if } i=j=0 \\ -1 & \text{if } i=j=1,2,3 \\ 0 & \text{otherwise} \end{cases}$ $\quad \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

If you go to another reference frame, the coordinates of the events are related by the Lorentz transformation. It can be shown (proof in HW) that Δs^2 is the same in the new system. It is a "Lorentz invariant."

We can also write the flat-space metric in spherical coordinates:

$$\Delta s^2 = (c\Delta t)^2 - (\Delta r)^2 - (r\Delta\theta)^2 - (r\sin\theta\,\Delta\varphi)^2$$

The metric coefficients change, but the spacetime is still "flat"

Differential geometry: Again, we can construct a curvature tensor out of the metric coefficients and their first derivatives. It vanishes in both cases.

> If there is gravity, we must be in a non-inertial (accelerating) reference frame. We can still write

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

As always, the actual metric coefficients depend on the choice of coordinate system. But GR asserts that the curvature tensor does not vanish. *

Equivalence Principle Revisited

If we transform to a freely-falling (inertial) reference frame, both the coordinates and the metric coefficients change. If we use Cartesian coordinates in the inertial frame, then the metric coefficients become η_{ij} (~~Euclidean~~ Minkowski) **locally** – it is not Euclidean everywhere.

Field Equations

The exact connection between mass and curvature is given by the field equations:

$$G_{\mu\nu} = \frac{-8\pi G}{c^4} T_{\mu\nu}$$

A non-linear equation, since the grav field itself acts as a source of mass/energy

$G_{\mu\nu}$ (Einstein tensor) is a combination of the curvature tensor and $g_{\mu\nu}$ itself.

$T_{\mu\nu}$ is the "stress-energy tensor". This describes all non-gravitational sources of mass and energy.

Schwarzschild Metric

Schwarzschild set $T_{\mu\nu} = 0$ and obtained the metric for the spacetime surrounding a point mass M located at the origin. The result is

$$\boxed{(\Delta s)^2 = \left(c\,\Delta t\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,\delta\varphi)^2}$$

Notice that the angular parts are the same as in flat space, reflecting spherical symmetry.

Addendum on Light

Recall that in special relativity, $\Delta s^2 = 0$. GR asserts that this remains true even in a non-inertial frame.

Consider a light ray propagating radially outward in the Schwarzschild metric. Here, $d\theta = d\varphi = 0$

so $$c\,dt\sqrt{1 - 2GM/rc^2} = \frac{dr}{\sqrt{1 - 2GM/rc^2}}$$

Thus, the time for light to propagate from r_1 to r_2 is

$$t = \int dt = \frac{1}{c}\int_{r_1}^{r_2} \frac{dr}{1 - \frac{2GM}{rc^2}}$$

NB: It doesn't matter when the light leaves r_1; the travel time is always the same; a consequence of the fact that the gravitational field is time-independent. This fact will prove to be important.