L4 - General Relativity II

Gravitational Redshift

Remember that Δs is an invariant. Under some circumstances (to be specified below), it may be interpreted as the proper time elapsing between two events occurring on the same object — and comoving with it.

Return to the Schwarzschild metric. Two pulses are emitted from the same location, a radius r_1 from the center. Since $\Delta r = \Delta\theta = \Delta\varphi = 0$, we have

$$\Delta\tau_1 = \Delta t_1 \sqrt{1 - \frac{2GM}{r_1 c^2}}$$

The same two ~~pluses~~ pulses are received at $r_2 > r_1$. The proper time interval for the arrival is

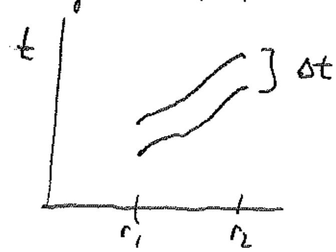
$$\Delta\tau_2 = \Delta t_2 \sqrt{1 - \frac{2GM}{r_2 c^2}}$$

If the pulses are part of a light wave, then $\nu_1 = \Delta\tau_1^{-1}$ etc.* So

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{1 - 2GM/r_1c^2}{1 - 2GM/r_2c^2}}\ \frac{\Delta t_1}{\Delta t_2}$$

Remember that the coordinate time t_{21} for each pulse to get from r_1 to r_2 is always the same. Thus, $\Delta t_1 = \Delta t_2$

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{1 - 2GM/r_1c^2}{1 - 2GM/r_2c^2}}$$



Suppose $r_2 = \infty$, then $r_1 \to r_0$

$$\boxed{\frac{\nu_\infty}{\nu_0} = \sqrt{1 - \frac{2GM}{r_0 c^2}}}$$

gravitational redshift

Light emitted from a massive object falls in frequency.

In other words, a clock deep within gravitational potential well appears to run more slowly.

Measuring the Redshift

For the Sun, $\frac{2GM}{Rc^2} = 3.8 \times 10^{-6}$

The effect is swamped by Doppler shifts arising from the motion of the Earth relative to the Sun ~~...~~ But this motion is well known & can be subtracted. The thermal agitation at the Sun's surface gives a Doppler shift, but this is a broadening, not a shift. The real problem is convection on the solar surface. This can even give a blueshift!

The situation is much better for a white dwarf. The most famous one is Sirius B, only 2.6 pc away.

Its mass is $M = 0.98\, M_\odot$

Measured $\left.\begin{array}{l} L = 0.027\, L_\odot \\ T_{eff} = 25{,}200\ K \end{array}\right\}$ using $L = 4\pi R^2 \sigma T_{eff}^4$ $\downarrow$ $R = 0.0086\, R_\odot$

Here, $\frac{2GM}{Rc^2} = 4.8 \times 10^{-4}$ Can we measure the redshift?

The above ratio is small enough that we can use $\frac{\nu_\infty}{\nu_0} \simeq 1 - \frac{GM}{Rc^2} = \frac{\lambda_0}{\lambda_\infty}$

Thus, $\frac{\lambda_\infty}{\lambda_0} \simeq 1 + \frac{GM}{Rc^2}$

The redshift is $\frac{\lambda_\infty - \lambda_0}{\lambda_0} = \frac{GM}{Rc^2} = 2.4 \times 10^{-4}$
(z)

This redshift is conventionally expressed as a velocity v through the (non-relativistic) formula

$$V = z \cdot c = 2.4 \times 10^{-4} \times 3 \times 10^5 \text{ km/s} = 72 \text{ km/s}$$

The best measured value (Barlow et al 2005) is 80 km/s ✓

Redshift: Weak-Field Limit

Let the light propagate only between r_1 and r_2. Both are in a weak field $\left(\frac{2GM}{Rc^2} \ll 1\right)$ and they are relatively close together: $r_2 - r_1 \equiv \Delta H \ll r_1$

$$\frac{\nu_2}{\nu_1} \simeq \frac{1 - \frac{GM}{r_1c^2}}{1 - \frac{GM}{r_2c^2}} \simeq \left(1 - \frac{GM}{r_1c^2}\right)\left(1 + \frac{GM}{r_2c^2}\right)$$

$$\simeq 1 - \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 1 - \frac{GM}{c^2}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

$$\simeq 1 - \frac{GM\Delta H}{c^2 r_1^2}$$

So $\boxed{\frac{\Delta\nu}{\nu_1} = \frac{\nu_2 - \nu_1}{\nu_1} = -\frac{GM\Delta H}{c^2 r_1^2}}$

$$= -\frac{g\Delta H}{c^2}$$

where $g \equiv \frac{GM}{r_1^2}$ is the surface gravity

We can understand this very crudely as follows: *

The energy of the photon is $E = h\nu + mgH$

where its effective mass is $mc^2 = h\nu_1 \longrightarrow m = \frac{h\nu_1}{c^2}$

Energy is conserved as it climbs in height, so

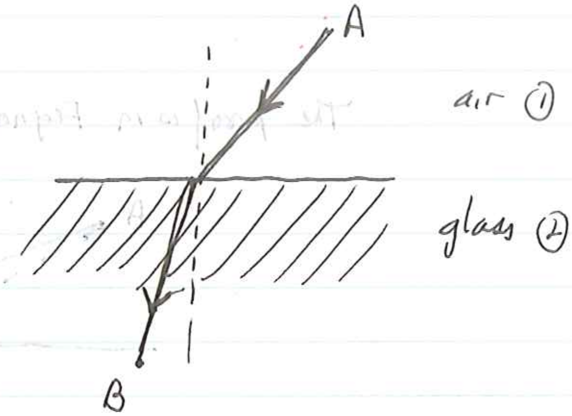
$$h\Delta\nu + mg\Delta H = 0 \longrightarrow \Delta\nu = -\frac{mg\Delta H}{h} = -\frac{h\nu_1}{c^2}\frac{g\Delta H}{h}$$

$$= -\frac{\nu_1 g \Delta H}{c^2}$$

$$\longrightarrow \boxed{\frac{\Delta\nu}{\nu_1} = -\frac{g\Delta H}{c^2}} \quad \checkmark$$

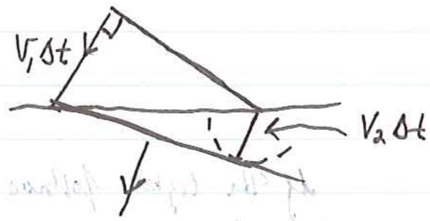
Not rigorous, since there is really no "gravitational P.E." for a photon.

Geodesics

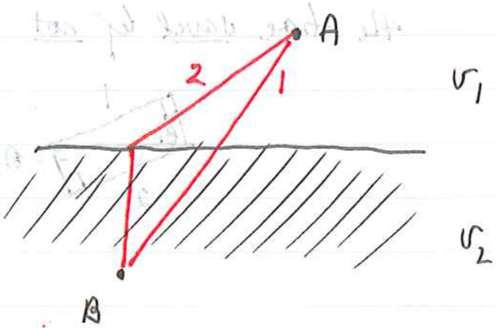


In optics, we have Snell's law. This is usually derived via the wave view, using a "Huygens construction."

A plane wave front hits the boundary, as shown, at $t=0$. A time Δt later, the left side of the top front hits the bdry. Over that time, the right side radiates a spherical wave. (Intermediate points radiate smaller spherical waves). Connecting the tangents, you get the new, tilted wavefront.



In relativity, we view light as a photon (particle). Can Snell's law still be explained? <u>YES</u>!



Fix the two points A and B. The path taken by the photon is that which <u>takes the least time</u> (Fermat's principle). For example, if $v_2 = v_1$, the path is clearly a straight line. But now a straight line (1) is ruled out, because too much time is consumed in the glass. Similarly, path 2 takes too much time in air. The true path obey's Snell's law.*

In other words - if you deviate from the true path even slightly, the total time ~~decreases~~ increases. The path is a "geodesic" — an <u>extremum</u> in the total time.

In GR, a ~~freely~~ freely falling particle travels through spacetime in such a way that the <u>total proper time</u> $\tau_{AB} = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ is an extremum.**

Light travels such that $\tau_{AB} = 0$

Classifying Spacetime Intervals

Returning to flat spacetime: $\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (c\Delta t)^2 - (\Delta r)^2$

* Suppose $(\Delta s)^2 > 0$ "timelike"

Then $\Delta r < c\Delta t$ Since nothing can travel faster than c, this interval is the possible path of a particle. The two events can be causally connected. There exists a reference frame in which the particle is stationary. In that frame, $\Delta s = c\Delta\tau$, by definition of proper time.

$$\boxed{\text{proper time: } \Delta\tau \equiv \frac{\Delta s}{c}}$$

* Suppose $(\Delta s)^2 < 0$ "spacelike"

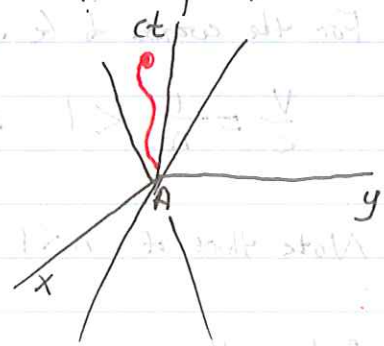
Then $\Delta r > c\Delta t$ This cannot represent the path of a particle. The two events cannot be causally connected. There exists a reference frame in which the two events are simultaneous.* In that frame, the distance btwn the events is spacial, & called the "proper distance."

$$\boxed{\text{proper distance: } \Delta L \equiv \sqrt{-(\Delta s)^2}}$$

* Suppose $(\Delta s)^2 = 0$ "null" This represents the path of a photon.

Worldlines

A light pulse is emitted at the origin & travels in (x,y) - a plane. We can picture its path in 3D - it traverses a light cone.



Now consider any event A at the origin.
If it is a flash on a material particle, that particle must have a spacetime path ("worldline") that lies within the light cone (timelike).

The interior of the upper light cone comprises all future events of A.
The interior of the lower " " " " past " " "

Everything outside the light cone is causally unconnected - "elsewhere"

All these concepts go over into GR.

1/15/12

Extra: <u>From GR to SR</u> (see also Weinberg, p 26–29)

Let $ds^2 = \eta_{ij}\, dx^i dx^j$ and demand that ds^2 is the same in another flat-space metric. Here
$$\eta_{ij} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

The L transformation is
$$(x')^i = \Lambda^i{}_j\, x^j$$
— I will find the $\Lambda^i{}_j$ (which is SR)

For invariance,
$$ds^2 = \underline{\eta_{ij}\, dx^i dx^j} = \eta_{kl}\,(dx')^k (dx')^l$$
$$= \eta_{kl}\,\Lambda^k{}_m dx^m\, \Lambda^l{}_n dx^n = \Lambda^k{}_m \Lambda^l{}_n\, \eta_{kl}\, dx^m dx^n$$
$$= \underline{\Lambda^k{}_i \Lambda^l{}_j\, \eta_{kl}\, dx^i dx^j}$$

Comparing the <u>underlined</u> expressions, we have $\boxed{\Lambda^k{}_i \Lambda^l{}_j\, \eta_{kl} = \eta_{ij}}$ (i)

* An observer O sees a particle <u>at rest</u> ($dx = 0$, $dt \neq 0$). Observer O′ sees it move at $-V$. That is, $dx'/dt' = -V$. The L transformation gives
$$(dx')^1 = \Lambda^1{}_j\, dx^j = \Lambda^1{}_0\, dt = -V dt'$$
$$(dx')^0 = dt' = \Lambda^0{}_0\, dt$$
Thus, $\Lambda^1{}_0 = -V\left(\frac{dt'}{dt}\right) = -V\Lambda^0{}_0$ $\qquad \boxed{\Lambda^1{}_0 = -V\Lambda^0{}_0}$

* An observer O sees a particle move at $+V \rightarrow dx = V dt$. The observer O′ sees it <u>at rest</u> ($dx' = 0$). The L transformation now gives
$$0 = \Lambda^1{}_0\, dt + \Lambda^1{}_1\, dx = -V\Lambda^0{}_0\, dt + \Lambda^1{}_1\, V dt$$
(use $c = 1$) Thus
$$\boxed{\Lambda^1{}_1 = \Lambda^0{}_0}$$

* Let $i = 1$, $j = 0$ in Eqtn (i):
$$\Lambda^k{}_1 \Lambda^l{}_0\, \eta_{kl} = \eta_{10} = 0 \rightarrow \Lambda^0{}_1 \Lambda^0{}_0 - \Lambda^1{}_1 \Lambda^1{}_0 = 0$$
Since $\Lambda^1{}_1 = \Lambda^0{}_0$, we have
$$\boxed{\Lambda^0{}_1 = \Lambda^1{}_0} \boxed{= -V\Lambda^0{}_0}$$

* Finally, let $i = j = 0$ in Eqtn (i):
$$\Lambda^k{}_0 \Lambda^l{}_0\, \eta_{kl} = \eta_{00} = +1 \rightarrow (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 = 1$$
$$(\Lambda^0{}_0)^2 - V^2 (\Lambda^0{}_0)^2 = 1$$
$$\boxed{(\Lambda^0{}_0)^2 = \frac{1}{1 - V^2} = \gamma^2}$$ (here $c = 1$)

– over –