L5: Black Holes I

Return to SR   Interlude: Momentum & Energy

From the HW problem, we know that $\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ is the same in every reference frame. We will write this as

$$\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

This is now an "invariant dot product." $dx^0 = c\,dt$ $dx^1 = dx$ etc

If the interval is timelike & describes a particle with velocity $\vec{v}$, then

$$c^2 d\tau^2 = c^2 dt^2 - v^2 dt^2 \longrightarrow \boxed{\frac{d\tau}{dt} = \sqrt{1 - v^2/c^2} = \frac{1}{\gamma}}$$

We also know that $\vec{p} = \gamma m \vec{v}$ and $E = \gamma m c^2$

I'll now show that the relationship

$$m^2c^4 = E^2 - p^2c^2$$

is just another "invariant dot product." First, note that

$$\vec{p} = \gamma m \frac{d\vec{r}}{dt} = m\frac{d\vec{r}}{d\tau} \qquad E = mc^2 \frac{dt}{d\tau}$$

We have, from above, $1 = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{1}{c}\frac{dx}{d\tau}\right)^2 - \left(\frac{1}{c}\frac{dy}{d\tau}\right)^2 - \left(\frac{1}{c}\frac{dz}{d\tau}\right)^2$

Multiply by m^2c^4: $\quad m^2c^4 = m^2c^4\gamma^2 - c^2 m^2\left(\frac{dx}{d\tau}\right)^2 - c^2 m^2\left(\frac{dy}{d\tau}\right)^2 - c^2 m^2\left(\frac{dz}{d\tau}\right)^2$

$$m^2c^4 = E^2 - c^2\left(p_x^2 + p_y^2 + p_z^2\right) = E^2 - p^2c^2 \quad \checkmark$$

And, we write this as $\underbrace{\boxed{m^2c^4 = \eta_{\mu\nu}\, p^\mu p^\nu}}_{\text{invariant dot product}}$ p^μ is the "momentum four-vector"

"Four-velocity" $u^\mu = (\gamma c, d\vec{r}/d\tau) = \gamma(c, \vec{v})$ ‖ $p^\mu = (E, \vec{p}c)$

$$c^2 = \eta_{\mu\nu}\, u^\mu u^\nu$$

Escape Speed

In Newtonian physics, we have for a particle leaving a body of mass M_* and size R.

$$\frac{1}{2}V^2 - \frac{GM_*}{r} = \frac{E}{m}$$

In order to escape, $v \to 0$ as $r \to \infty$ $\rightarrow$ $V_{esc} = \sqrt{\frac{2GM_*}{R}}$

Q: What is R such that $v = c$?

A: $\boxed{R_{crit} = \frac{2GM_*}{c^2}}$

According to Newton, a body this small has such strong gravity that even light has trouble escaping.

It turns out that the same is true of GR $\rightarrow$ $\boxed{R_S \equiv \frac{2GM_*}{c^2}}$ and defines the "Schwarzschild radius."

for $M = 1\,M_\odot$
$R_S = 3$ km!

Falling In

Return to the Schwarzschild metric, & make $d\theta = d\phi = 0$ for simplicity: If the interval is timelike,

$$d\tau^2 = \left(dt\sqrt{1 - R_S/r}\right)^2 - \left(\frac{1}{c}\frac{dr}{\sqrt{1 - R_S/r}}\right)^2$$

definition of black hole

Suppose I am falling toward a mass whose physical radius is less than R_S. (a "black hole").

Remember that $d\tau$ is my proper time. Since I am freely falling, I feel no gravity, & my clock ticks normally. Thus, $d\tau$ is a finite number.

But, as $r \to R_S$ dr has to shrink to zero to keep $d\tau$ finite. To an external observer, therefore, I come to a stop at $r = R_S$!

singularity

From my perspective, I keep on going to $r = 0$, which is the singularity. No one knows what happens there.

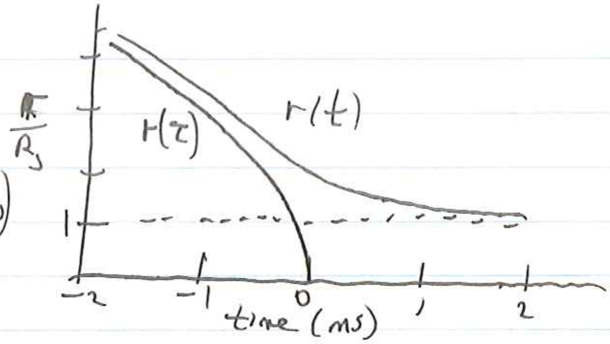
static limit

The radius $r = R_S$ is also called the static limit. Suppose $r < R_S$ and r is fixed. Then $d\tau^2 = dt^2\left(1 - R_S/r\right) < 0$ which is impossible. The astronaut agrees: once he crosses $r = R_S$, he must fall to $r = 0$

frozen star

In summary, we plot $r(\underline{t})$ & $r(\underline{\tau})$:

(for $M_* = 10\,M_\odot$)



When a BH forms, the gas must similarly pile up. In other languages, the object is called a "frozen star."

Climbing Out

The particle with the best chance of escaping is, of course, the photon.

Q: How long does it take for a radially directed photon to go from r_1 to r_2?

A: Now the interval is "null": $0 = \left(dt\sqrt{1-R_s/r}\right)^2 - \left(\frac{1}{c}\frac{dr}{\sqrt{1-R_s/r}}\right)^2$

$$\text{So}\quad t_{12} = \int dt = \frac{1}{c}\int_{r_1}^{r_2}\frac{dr}{1-R_s/r} = \frac{R_s}{c}\int_1^2\frac{dx}{1-1/x}\qquad x \equiv \frac{r}{R_s}$$

$$= \frac{R_s}{c}\int_1^2\frac{x\,dx}{x-1} = \frac{R_s}{c}\int_1^2\frac{(x-1)\,dx}{(x-1)} + \frac{R_s}{c}\int_1^2\frac{dx}{x-1}$$

$$= \frac{R_s}{c}\left[\frac{r_2}{R_s}-\frac{r_1}{R_s}\right] + \frac{R_s}{c}\ln\left(\frac{r_2-R_s}{r_1-R_s}\right) = \boxed{\frac{r_2-r_1}{c} + \frac{R_s}{c}\ln\left(\frac{r_2-R_s}{r_1-R_s}\right)}$$

If $r_1 = R_s$, $t_{12} = \infty$: it takes the photon forever to escape!

event horizon

This is why the object is a "black hole" – even light cannot escape. So $r = R_s$ is also called the "event horizon." ∴ can know nothing of what occurs inside.

COSMIC CENSORSHIP (hypothesis): ∄ no "naked singularities", ie, unclothed by event horizon.

Incidentally, the above null interval shows that the coordinate speed of a photon is

$$v_{photon} = \frac{dr}{dt} = c\left(1 - R_s/r\right)$$

So even a photon ~~approaching~~ falling into $r = R_s$ seems to creep to a halt.

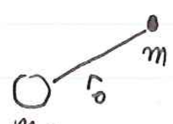
Redshift

Recall that $\frac{\nu_\infty}{\nu_0} = \sqrt{1 - \frac{2GM}{r_0 c^2}} = \sqrt{1 - \frac{R_s}{r_0}}$ where r_0 is the starting radius

As matter falls into a BH, photons become increasingly redshifted, until $\nu_\infty \to 0$.
So $r = R_s$ is also (!) the "∞ redshift surface."

Orbiting a BH

Review: Newtonian circular

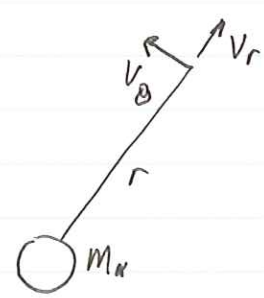


$$\frac{GM_* m}{r_0^2} = \frac{mV^2}{r_0} \to r_0 = \frac{GM_*}{V^2}$$

Recast in terms of $L = mVr_0$ | $r_0 = \frac{GM_* m^2 r_0^2}{m^2 V^2 r_0^2} = \frac{GM_* m^2 r_0^2}{L^2}$

So that $\boxed{r_0 = \left(\frac{L}{m}\right)^2 \frac{1}{GM_*} = \frac{\ell^2}{GM_*}}$ where $\ell \equiv L/m$

Review: Newtonian general orbit



Since the force exerts no torque, $L = mV_\theta r$ is still a constant. Write as $\ell = \omega r^2$
where $\omega \equiv V_\theta / r$

NB: there are (two) conserved quantities

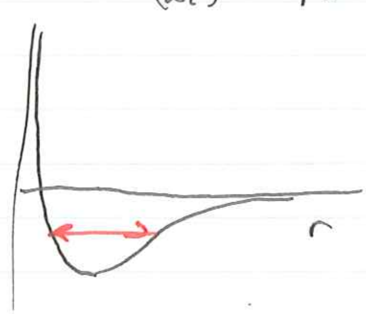
Also conserved is the energy per unit mass:

$$\mathcal{E} \equiv \frac{E}{m} = \frac{1}{2}V^2 - \frac{GM_*}{r}$$

But $V^2 = \left(\frac{dr}{dt}\right)^2 + \omega^2 r^2 = \left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{r^2}$

$$\text{So } \mathcal{E} = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \left[-\frac{GM_*}{r} + \frac{\ell^2}{2r^2}\right]$$

The [ ] term is the "effective potential" V_{eff}
Particle can be trapped (an ellipse).



At circular orbit, $\frac{dV_{eff}}{dr} = 0 = \frac{+GM_*}{r_0^2} - \frac{\ell^2}{r_0^3}$

$$\boxed{r_0 = \frac{\ell^2}{GM_*}} \checkmark$$

Use a similar argument in GR: $m^2c^4 = g_{\mu\nu}p^\mu p^\nu \longrightarrow$

$$\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 = \mathcal{E}^2 - \left(1-\frac{R_s}{r}\right)\left(1+\frac{\ell^2}{c^2r^2}\right)$$

(τ, not t)

Here, there are again (2) constants: $\mathcal{E} \equiv$ "energy at infinity" per rest mass energy
ℓ = "total angular momentum" per unit rest mass

This equation is of the form

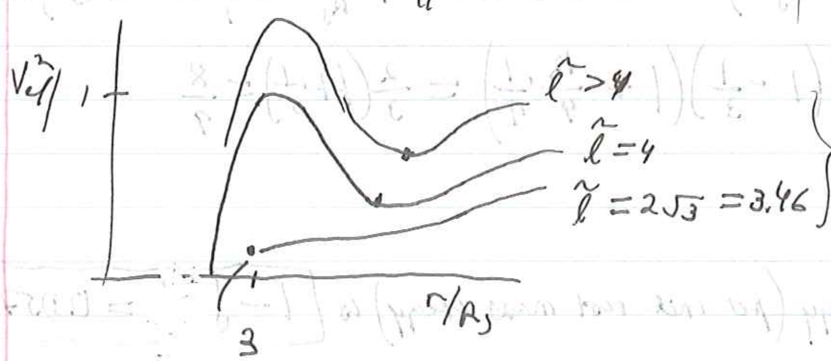
$$\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 = \mathcal{E}^2 - V_{eff}^2(r), \quad V_{eff}^2 = \left(1-\frac{R_s}{r}\right)\left(1+\frac{\ell^2}{c^2r^2}\right)$$

which can be written as

$$V_{eff}^2 = \left(1-\frac{R_s}{r}\right)\left[1+\frac{\tilde{\ell}^2}{4}\left(\frac{R_s}{r}\right)^2\right] \quad \left[\tilde{\ell} \equiv \frac{\ell c}{GM_*}\right]^*$$

As $r\to\infty$, $V_{eff}\to 1$ (not zero!) Big View:

A closer view near $V_{eff} = 1$:



The "pit" is the new, GR feature.

- A particle with $\tilde{\ell} = 0$ must plunge to $r=0$; no bound orbits
- For $\tilde{\ell} > 4$, the particle can come in from ∞, bounce, + then go back out OR it can just plunge in, depending on $\mathcal{E}$. There are bound orbits, as well.
- For $2\sqrt{3} < \tilde{\ell} < 4$, there are bound orbits, $\boxed{\text{or}}$ a (high-$\mathcal{E}$) particle gets pulled in.
- The smallest stable orbit is at $r = 3R_s$.

Mercury

All orbits for which $r_0 >> R_s$ are almost keplerian, except that the periastron slowly precesses
— Mercury 43"/century

Uniformly Accelerated Particle

It is sometimes said that special relativity cannot treat accelerating motion. Untrue! Consider a particle which is uniformly accelerating (at g) in its own frame

Let S' be the frame in which the particle initially has $V_x' = 0$. A short time later ($\Delta t' = \Delta\tau$), the particle has speed $V_x' = g\Delta\tau$ in this frame.

What is the velocity in a stationary frame S? Use $\displaystyle V_x = \frac{V_x' + V}{1 + \frac{V}{c^2}V_x'}$

When $\tau = 0$, $\quad V_x = (0 + V)/(1+0) = V$

When $\tau = \Delta\tau$, $\quad V_x = (g\Delta\tau + V)\Big/\left(1 + \frac{Vg\Delta\tau}{c^2}\right) \doteq V + g\left(1 - \frac{V^2}{c^2}\right)\Delta\tau + O(\Delta\tau^2)$

Thus $\quad \Delta V_x = g\left(1 - \frac{V^2}{c^2}\right)\Delta\tau \qquad$ But $\begin{bmatrix} \Delta V_x = \Delta V, \text{ since } V = V_x \\ \text{Also } \Delta t = \gamma\,\Delta\tau \text{ (time dilation)} \end{bmatrix}$

$$\frac{dV}{1 - V^2/c^2} = g\,d\tau \longrightarrow \frac{d\beta}{1-\beta^2} = \frac{g}{c}\,d\tau \qquad \text{where } \beta \equiv V/c$$

Use $\displaystyle \frac{1}{1-\beta^2} = \frac{1}{2}\left[\frac{1}{1-\beta} + \frac{1}{1+\beta}\right] \longrightarrow \ln\left(\frac{1+\beta}{1-\beta}\right) = \frac{2g\tau}{c}$ assuming $\beta = 0$ at $\tau = 0$

$$\frac{1+\beta}{1-\beta} = e^{2g\tau/c} \longrightarrow \beta = \frac{e^{2g\tau/c} - 1}{e^{2g\tau/c} + 1} = \frac{e^{g\tau/c} - e^{-g\tau/c}}{e^{g\tau/c} + e^{-g\tau/c}} = \tanh\left(\frac{g\tau}{c}\right)$$

What is γ? $\displaystyle \frac{1}{\gamma^2} = 1 - \beta^2 = 1 - \tanh^2\left(\frac{g\tau}{c}\right) = \operatorname{sech}^2\left(\frac{g\tau}{c}\right) \longrightarrow \gamma = \cosh\left(\frac{g\tau}{c}\right)$

$$\beta = \frac{1}{c}V = \frac{1}{c}\frac{dx}{dt} = \frac{1}{c}\frac{d\tau}{dt}\cdot\frac{dx}{d\tau} = \frac{1}{\gamma c}\frac{dx}{d\tau} = \tanh\left(\frac{g\tau}{c}\right)$$

So $\displaystyle \boxed{\frac{1}{c}\frac{dx}{d\tau}} = \gamma\tanh\left(\frac{g\tau}{c}\right) = \cosh\left(\frac{g\tau}{c}\right)\tanh\left(\frac{g\tau}{c}\right) = \boxed{\sinh\left(\frac{g\tau}{c}\right)}$

and $\displaystyle \boxed{\frac{dt}{d\tau}} = \gamma = \boxed{\cosh\left(\frac{g\tau}{c}\right)}$

Integrating,

$$\boxed{x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right)\ ; \quad t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right)}$$

Notice: $\displaystyle x^2 - c^2t^2 = \frac{c^4}{g^2}$

$$\boxed{x = \sqrt{\frac{c^4}{g^2} + c^2t^2}}$$

world line: